The number of $L_{\infty\kappa}$-equivalent non-isomorphic models for κ weakly compact

Saharon Shelah* Pauli Väisänen†

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Abstract

For a cardinal κ and a model $\mathcal{M}$ of cardinality κ let $\mathrm{No}(\mathcal{M})$ denote the number of non-isomorphic models of cardinality κ which are $L_{\infty\kappa}$-equivalent to $\mathcal{M}$. In [She82] Shelah established that when κ is a weakly compact cardinal and $\mu \leq \kappa$ is a nonzero cardinal, there exists a model $\mathcal{M}$ of cardinality κ with $\mathrm{No}(\mathcal{M}) = \mu$. We prove here that if κ is a weakly compact cardinal, the question of the possible values of $\mathrm{No}(\mathcal{M})$ for models $\mathcal{M}$ of cardinality κ is equivalent to the question of the possible numbers of equivalence classes of equivalence relations which are Σ^1_1-definable over V_κ. In [SVa] we proved that, consistent wise, the possible numbers of equivalence classes of Σ^1_1-equivalence relations can be completely controlled under the singular cardinal hypothesis. These results settle the problem of the possible values of $\mathrm{No}(\mathcal{M})$ for models of weakly compact cardinality, provided that the singular cardinal hypothesis holds. [1]

1 Introduction

Suppose κ is a cardinal and $\mathcal{M}$ is a model of cardinality κ. Let $\mathrm{No}(\mathcal{M})$ denote the number of non-isomorphic models of cardinality κ which are

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elementary equivalent to $\mathcal{M}$ over the infinitary language $L_{\infty\kappa}$. We study the possible values of $\mathrm{No}(\mathcal{M})$ for different models $\mathcal{M}$.

When $\mathcal{M}$ is countable, $\mathrm{No}(\mathcal{M}) = 1$ by [Sco65]. This result extends to all structures of singular cardinality λ provided that λ is of countable cofinality [Cha68]. The case $\mathcal{M}$ is of singular cardinality λ with uncountable cofinality κ was first treated in [She85] and later on in [She86]. In these papers Shelah showed that if $\kappa > \aleph_0$, $\theta^\kappa < \lambda$ for every $\theta < \lambda$, and $0 < \mu < \lambda$ or $\mu = \lambda^\kappa$, then $\mathrm{No}(\mathcal{M}) = \mu$ for some model $\mathcal{M}$ of cardinality λ. In the paper [SVb] of the authors the singular case is revisited, and particularly, it is established, under the same assumptions as above, that the values μ with $\lambda \leq \mu < \lambda^\kappa$ are possible for $\mathrm{No}(\mathcal{M})$ with $\mathcal{M}$ of cardinality λ.

If $V = L$, κ is an uncountable regular cardinal which is not weakly compact, and $\mathcal{M}$ is a model of cardinality κ, then $\mathrm{No}(\mathcal{M}) \in \{1, 2^\kappa\}$ [She81]. For $\kappa = \aleph_1$ this result was first proved in [Pal77a]. The values $\mathrm{No}(\mathcal{M}) \in \{\aleph_0, \aleph_1\}$ for a model of cardinality $\aleph_1$ are consistent with ZFC + GCH as noted in [She81]. All the nonzero finite values of $\mathrm{No}(\mathcal{M})$ for models of cardinality $\aleph_1$ are proved to be consistent with ZFC + GCH in [SVc].

When κ is a weakly compact cardinal and μ is a nonzero cardinal $\leq \kappa$ there is a model $\mathcal{M}$ of cardinality κ with $\mathrm{No}(\mathcal{M}) = \mu$ [She82]. In the present paper we show that when κ is a weakly compact cardinal, the possible values of $\mathrm{No}(\mathcal{M})$ for models of cardinality κ depends only on the possible numbers of equivalence classes of equivalence relations which are Σ^1_1-definable over V_κ as follows: for some first order sentence ϕ in the vocabulary $\{\in, R_0, R_1, R_2, R_3\}$ and a subset P of V_κ, it is the case that for all $s, t \in {}^\kappa 2$,

$$s \sim t \text{ iff for some } r \in {}^\kappa 2 \ \langle V_\kappa, \in, P, s, t, r\rangle \models \phi$$

where P, r, s, and t are the interpretations of the symbols R_0, R_1, R_2, and R_3 respectively. More precisely, we prove the following theorem.

Theorem 1 *When κ is a weakly compact cardinal, the following two conditions are equivalent for every nonzero cardinal μ:*

a) *there is an equivalence relation on ${}^\kappa 2$ which is Σ^1_1-definable over V_κ and has exactly μ different equivalence classes;*

b) $\mathrm{No}(\mathcal{M}) = \mu$ *for some model $\mathcal{M}$ of cardinality κ.*

In the paper [SVa] we proved that for every nonzero cardinal $\mu \in \kappa \cup \{\kappa, \kappa^+, 2^\kappa\}$ there is always a Σ^1_1-equivalence relation (as defined above)

with exactly μ different equivalence classes. Moreover, consistent wise, one can completely control the possible numbers of equivalence classes of Σ^1_1-equivalence relations provided that the singular cardinal hypothesis holds [SVa, Theorem 1]. It follows that, the question of possible value of $\mathrm{No}(\mathcal{M})$ is completely solved, when $\mathcal{M}$ is of weakly compact cardinality and the singular cardinal hypothesis holds. Again more formally, the conclusion will be the following.

Conclusion 1.1 *Suppose that the following conditions are satisfied:*

> *κ is a weakly compact cardinal;*
>
> *κ remains a weakly compact cardinal in the standard Cohen forcing adding a new subset of κ;*
>
> *the singular cardinal hypothesis holds;*
>
> *$\lambda > \kappa^+$ is a cardinal with $\lambda^\kappa = \lambda$;*
>
> *Ω is a set of cardinals between κ^+ and λ (possibly empty), which is closed under unions of $\leq \kappa$-many cardinals and products of $< \kappa$-many cardinals.*

Then there is a forcing extension where there are no new sets of cardinality $< \kappa$, all cardinals and cofinalities are preserved, κ remains a weakly compact cardinal, $2^\kappa = \lambda$, and for all cardinals μ, there exists a model $\mathcal{M}$ of cardinality κ with $\mathrm{No}(\mathcal{M}) = \mu$ if and only if μ is a nonzero cardinal $\leq \kappa^+$ or μ is in $\Omega \cup \{2^\kappa\}$.

Remark. When κ is a weakly compact cardinal, it is possible to have, using the upward Easton forcing, a generic extension where κ is still a weakly compact cardinal and κ remains weakly compact in the Cohen forcing adding a subset of κ (Silver). The forcing needed in the conclusion is the ordinary way to add Kurepa trees of height κ with μ-many κ-branches through them, for all $\mu \in \Omega$. As noted in [SVa, Fact 5.1], this forcing is locally κ Cohen, and therefore, κ remains a weakly compact cardinal in the composite forcing of the upward Easton forcing and the addition of new Kurepa trees.

Note also, that the closure properties mentioned in the conclusion are necessary by the fact that the possible numbers of equivalence classes of Σ^1_1-equivalence relations are always closed under unions of length $\leq \kappa$ and products of length $< \kappa$, see [SVa, Lemma 3.4].

There are three parts in the paper. First in Section 2 we recall a definition of a Ehrenfeucht-Fraïssé-game $\mathrm{EF}_{\kappa;\lambda}(\mathcal{M},\mathcal{N})$ generalizing the elementary equivalence between two models over an infinitary language $L_{\infty\kappa}$. In addition to that, we show that if $\mathcal{M}$ is a model of cardinality κ, then there is a Σ^1_1-equivalence relation with exactly $\mathrm{No}(\mathcal{M})$ different equivalence classes. We also note how this connection extends to even more strongly notions of equivalence between models.

The last two sections are dedicated to the other half of the proof of the theorem, namely to the proof that the existence of a Σ^1_1-equivalence relation with μ-many equivalence classes implies the existence of a model $\mathcal{M}$ with $\mathrm{card}(\mathcal{M}) = \kappa$ and $\mathrm{No}(\mathcal{M}) = \mu$ (Lemma 4.8 at the end of Section 4). First in Section 3 we define a special family of functions which is used to build models in the last section. This part might feel quite technical. However, it is elementary and the fundamental idea of the construction is from [She82]. The reader may even skip all the lemmas of this section in the first reading, and return to those when they are referred from the last section.

The content of Section 4 is as follows. Assuming that κ is strongly inaccessible and ϕ defines a Σ^1_1-equivalence relation $\sim_{\phi,P}$ on ${}^{\kappa}2$ with a parameter $P \subseteq V_\kappa$ we construct models $\mathcal{M}_t$ for $t \in {}^{\kappa}2$ satisfying that

> the models are of cardinality κ and they have a common vocabulary ρ consisting of κ many relation symbols each of having arity $< \kappa$;
>
> all the models are pairwise $L_{\infty\kappa}$-equivalent, and even more, they are pairwise $M_{\infty\kappa;\lambda}$-equivalent for any previously fixed regular cardinal $\lambda < \kappa$ (see Definition 2.1);
>
> for all $s, t \in {}^{\kappa}2$, the models $\mathcal{M}_s$ and $\mathcal{M}_t$ are isomorphic if, and only if, s and t are equivalent with respect to $\sim_{\phi,P}$.

Furthermore, when κ is a weakly compact cardinal the models satisfy the additional property that

> if a model $\mathcal{N}$ has vocabulary ρ, $\mathcal{N}$ is of cardinality κ, and $\mathcal{N}$ is $L_{\infty\kappa}$-equivalent to one (all) of the models $\mathcal{M}_t$, $t \in {}^{\kappa}2$, then $\mathcal{N}$ is isomorphic to $\mathcal{M}_s$ for some $s \in {}^{\kappa}2$.

This is the main difference between the strongly inaccessible non-weakly compact case and the weakly compact case: the Π^1_1-indescribability property of a weakly compact cardinal κ ensures that the "isomorphism type" of any model $\mathcal{N}$ with domain κ is already determined by the isomorphism types of the bounded parts $\mathcal{N}\restriction\alpha$, $\alpha < \kappa$, alone (see the proof of Lemma 4.7).

2 Preliminaries

Definition 2.1 *Suppose μ is a cardinal and λ is a infinite regular cardinal. Let $\mathcal{M}$ and $\mathcal{N}$ be models of a common relational vocabulary. The Ehrenfeucht-Fraïssé-game $\mathrm{EF}_{\mu;\lambda}(\mathcal{M},\mathcal{N})$ is defined as follows. The game has two players, $\forall$ and $\exists$, and a play of the game continues for at most λ rounds. On round $i < \lambda$ player $\forall$ first chooses $X_i \in \{\mathcal{M},\mathcal{N}\}$ and $A_i \subseteq X_i$ of cardinality $< \mu$. Then $\exists$ replies with a partial isomorphism p_i such that*

$$\mathrm{dom}(p_i) \subseteq \mathcal{M},\ \mathrm{ran}(p_i) \subseteq \mathcal{N},\ \textstyle\bigcup_{j<i} p_j \subseteq p_i, \text{ and}$$

$$A_i \subseteq \mathrm{dom}(p_i) \text{ if } X_i = \mathcal{M}, \text{ and } A_i \subseteq \mathrm{ran}(p_i) \text{ otherwise.}$$

Player $\exists$ wins a play if the play lasts λ many rounds, and otherwise, $\forall$ wins the play. We write $\mathcal{M} \equiv_{\infty\kappa;\lambda} \mathcal{N}$ when $\exists$ has a winning strategy in $\mathrm{EF}_{\kappa;\lambda}(\mathcal{M},\mathcal{N})$.

Let $\mathcal{M}$ and $\mathcal{N}$ be models of a common relational vocabulary and κ be a cardinal. The game $\mathrm{EF}_{\kappa;\omega}(\mathcal{M},\mathcal{N})$ is the usual Ehrenfeucht-Fraïssé-game of length ω which characterizes the existence of a nonempty family of partial isomorphism with the "fewer than κ at the time back-and-forth property". If $\mathcal{M}$ and $\mathcal{N}$ satisfy the same sentences of the infinitary language $L_{\infty\kappa}$, we write $\mathcal{M} \equiv_{\infty\kappa} \mathcal{N}$. By the Karp's theorem [Kar65] player $\exists$ has a winning strategy in $\mathrm{EF}_{\kappa;\omega}(\mathcal{M},\mathcal{N})$ if, and only if, $\mathcal{M} \equiv_{\infty\kappa} \mathcal{N}$. So the game $\mathrm{EF}_{\kappa;\lambda}(\mathcal{M},\mathcal{N})$, for an infinite regular cardinal $\lambda < \kappa$, is a generalized version of the "fewer than κ at the time back-and-forth property". There are so-called infinitely deep languages $M_{\infty\kappa;\lambda}$ with the property that $\mathcal{M} \equiv_{\infty\kappa;\lambda} \mathcal{N}$ if, and only if, $\mathcal{M}$ and $\mathcal{N}$ satisfy the same sentences of $M_{\infty\kappa;\lambda}$ [Hyt90, Kar84, Oik97].

For a model $\mathcal{N}$ we let $\mathrm{card}(\mathcal{N})$ denote the cardinality of the universe of $\mathcal{N}$. For any model $\mathcal{M}$ of cardinality κ and a regular cardinal $\aleph_0 \le \lambda < \kappa$, we define $\mathrm{No}_\lambda(\mathcal{M})$ to be the cardinality of the set

$$\big\{\mathcal{N}/\!\cong\ \mid\ \mathrm{card}(\mathcal{N}) = \kappa \text{ and } \mathcal{N} \equiv_{\infty\kappa;\lambda} \mathcal{M}\big\},$$

where $\mathcal{N}/\!\cong$ is the equivalence class of $\mathcal{N}$ under the isomorphism relation.

For all sets X of ordinals, the ordinal $\sup\{\alpha + 1 \mid \alpha \in X\}$ is abbreviated by $\sup^+(X)$. For all sequences $\bar{\alpha} = \langle \alpha_i \mid i < \theta \rangle$ of ordinals, we denote $\sup^+\big(\{\alpha_i \mid i < \theta\}\big)$ by $\sup^+(\bar{\alpha})$, and we abbreviate the sequence $\langle f(\alpha_i) \mid i < \theta \rangle$ by $f(\bar{\alpha})$. For a regular cardinal κ and a subset S of κ, S is

called stationary if for every closed and unbounded subset C of κ, $S \cap C$ is nonempty.

Next we recall the definition of an equivalence relation which is Σ^1_1-definable over the set $H(\kappa)$ of all sets hereditarily of cardinality $< \kappa$ from [SVa, Definition 3.1]. In this paper κ will be a strongly inaccessible cardinal and so $H(\kappa)$ equals to the set V_κ of all sets having rank $< \kappa$. It will be more convenient to use V_κ instead of $H(\kappa)$ here, especially, when we consider elementary submodels of V_κ, and also, when we apply Π^1_1-indescribability property for κ weakly compact.

Definition 2.2 *Suppose κ is a strongly inaccessible cardinal. We say that ϕ defines a Σ^1_1-equivalence relation $\sim_{\phi,P}$ on ${}^\kappa 2$ with a parameter $P \subseteq V_\kappa$ when*

a) *ϕ is a first order sentence in a vocabulary consisting of $\in$, one unary relation symbol R_0, and binary relation symbols R_1, R_2, and R_3;*

b) *the following definition gives an equivalence relation on ${}^\kappa 2$: for all $s, t \in {}^\kappa 2$*

$$s \sim_{\phi,P} t \textit{ iff } \langle V_\kappa, \in, P, s, t, r\rangle \models \phi \textit{ for some } r \in {}^\kappa 2,$$

where P, s, t, and r are the interpretations of the symbols R_0, R_1, R_2, and R_3 respectively.

We say that there exists a Σ^1_1-equivalence relation on ${}^\kappa 2$ having μ many different equivalent classes when there is some sentence ϕ and a parameter $P \subseteq V_\kappa$ such that ϕ defines a Σ^1_1-equivalence relation $\sim_{\phi,P}$ on ${}^\kappa 2$ with the parameter P and $\mathrm{card}\big(\{f/{\sim_{\phi,P}} \mid f \in {}^\kappa 2\}\big) = \mu$.

Lemma 2.3 *If κ is a strongly inaccessible cardinal and $\mathcal{M}$ is a model of cardinality κ, then there is a Σ^1_1-equivalence relation on ${}^\kappa 2$ such that the number of different equivalence classes of it is* $\mathrm{No}(\mathcal{M})$.

Proof. For a function π having domain κ and a binary relation R, we let $\pi(R)$ denote the set $\{\pi(\xi) \mid \text{for some } \xi < \kappa, \langle \xi, 1\rangle \in R\}$. For every $n < \omega$, we write π_n for a fixed definable bijection from κ onto $\{\langle \alpha_1, \ldots, \alpha_n\rangle \mid \alpha_1, \ldots, \alpha_n \in \kappa\}$. We may assume that the domain of $\mathcal{M}$ is κ and its vocabulary consists of one relation symbol Q of finite arity n. For a binary relation R let $\mathcal{N}(R)$ be the model having domain κ and interpretation

$\pi_n(R)$ for the relation symbol Q. By the inaccessibility of κ, let ρ be a bijection from κ onto V_κ. For a binary relation R let $\tau_1(R)$ be the set $\{\rho(\xi) \mid \xi < \kappa$ is a successor ordinal and $\langle \xi, 1\rangle \in R\}$ and $\tau_2(R)$ be the set $\{\rho(\xi) \mid \xi < \kappa$ is a limit ordinal and $\langle \xi, 1\rangle \in R\}$.

Since for all models $\mathcal{N}$, the game $\mathrm{EF}_{\kappa;\omega}(\mathcal{M},\mathcal{N})$ is determined, the condition $\mathcal{M} \not\equiv_{\infty\kappa} \mathcal{N}$ is equivalent to that player $\forall$ has a winning strategy in $\mathrm{EF}_{\kappa;\omega}(\mathcal{M},\mathcal{N})$. Therefore, using the interpretation $Q^{\mathcal{M}}$ and the bijection ρ as a parameter, the sentence $\phi(R_0, R_1, R_2, R_3)$ saying

> "($\tau_1(R_3)$ is a winning strategy for player $\forall$ in $\mathrm{EF}_{\kappa;\omega}(\mathcal{M},\mathcal{N}(R_1))$
> and $\tau_2(R_3)$ is a winning strategy for player $\forall$ in $\mathrm{EF}_{\kappa;\omega}(\mathcal{M},\mathcal{N}(R_2))$
>) or ($\pi_2(R_3)$ is an isomorphism between $\mathcal{N}(R_1)$ and $\mathcal{N}(R_2)$)"

defines a Σ^1_1-equivalence relation on ${}^\kappa 2$. This definition is as wanted in the claim, except that when $\mathrm{No}(\mathcal{M})$ is finite the definition gives one extra class. That can be avoided by obvious changes to the definition. $\blacksquare_{2.3}$

Remark. As noted in [SVa, Section 5], the theorem on the possible numbers of equivalence classes of Σ^1_1-equivalence relations directly extends to equivalence relations which are definable over V_κ using a subset of V_κ as a parameter and a sentence which is a Boolean combination of a sentence containing one second order existential quantifier (Σ^1_1-sentence) and a sentence containing one second order universal quantifier (Π^1_1-sentence). In the proof of Lemma 2.3 above we needed the fact that the game $\mathrm{EF}_{\kappa;\omega}(\mathcal{M},\mathcal{N})$ is determined to find a Σ^1_1-sentence which says "if the models are equivalent then ... ", i.e., "either the models are nonequivalent or ... ". But there is a Π^1_1-sentence saying "there is no winning strategy for player $\exists$ in the game ... ". Hence, after we have proved that existence of a Σ^1_1-equivalence relation with μ-classes implies existence of a model $\mathcal{M}$ with $\mathrm{No}(\mathcal{M}) = \mu$ (after the next two sections), we can conclude: consistent wise, the possible values of $\mathrm{No}(\mathcal{M})$ for $\mathcal{M}$ of weakly compact cardinality κ might be exactly as wanted, and moreover, the possible values of $\mathrm{No}_\lambda(\mathcal{M})$, for $\mathcal{M}$ of cardinality κ and $\lambda < \kappa$ any regular cardinal, coincide with the possible values of $\mathrm{No}_\omega(\mathcal{M})$.

3 The family of functions

Throughout the next two sections κ is a strongly inaccessible cardinal, i.e., a regular limit cardinal satisfying $2^\mu < \kappa$ for all $\mu < \kappa$, and λ is a fixed regular cardinal below κ.

In this section we define a family of functions which will be used to build the models $\mathcal{M}_t$, $t \in {}^{\kappa}2$ (Definition 4.1). There is a similar idea in [She82], however, this time we want the functions to satisfy some additional properties. Hence the definition of the family will be a little bit more complicated. Particularly, we shall first define a special tree (Definition 3.5) which will be only a steering apparatus in the construction of the family of functions itself (Definition 3.9).

To make our models strongly equivalent we shall guarantee that for every pair $\mathcal{M}_s$ and $\mathcal{M}_t$, $s, t \in {}^{\kappa}2$, a certain subfamily of all the functions will form a winning strategy for player $\exists$ in the game $\mathrm{EF}_{\kappa;\lambda}(\mathcal{M}_s, \mathcal{M}_t)$ (Definition 3.4). So we shall need a "stronger extension property" for the functions than was needed in [She82].

Most importantly, we want that a pair $\mathcal{M}_s$ and $\mathcal{M}_t$, $s, t \in {}^{\kappa}2$, of models are isomorphic if, and only if, the corresponding indices s and t are equivalent with respect to some previously fixed Σ^1_1-equivalence relation (Lemma 4.4). Hence we have to code information about the equivalence relation into the family of functions (Definition 3.2 and Definition 3.9).

Henceforth ϕ denotes a sentence which defines a Σ^1_1-equivalence relation $\sim_{\phi,P}$ on ${}^{\kappa}2$ with a parameter $P \subseteq V_\kappa$ (see Definition 2.2). Without loss of generality we may assume that for all $s, t, r \in {}^{\kappa}2$,

$$\langle V_\kappa, \in, P, s, t, r\rangle \models \phi \text{ iff } \langle V_\kappa, \in, P, t, s, r\rangle \models \phi. \tag{1}$$

Furthermore, we may assume that for all $s, t \in {}^{\kappa}2$,

$$\text{if } s(\beth_\alpha) = t(\beth_\alpha) \text{ for every } \alpha < \kappa \text{ then } s \sim_{\phi,P} t. \tag{2}$$

Definition 3.1 *Let* $T[0]$ *be* $\{\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle\}$ *and for every nonzero* $\alpha < \kappa$ *define*

$$T[\alpha] = \{\langle \eta, \nu, \tau, C\rangle \mid \eta, \nu, \tau \in {}^{(\beth_\alpha)}2, \eta \neq \nu, \textit{and } C \textit{ is a closed subset of } \alpha\},$$

$T[<\alpha] = \bigcup_{\beta<\alpha} T[\beta]$, *and* $T = \bigcup_{\alpha<\kappa} T[\alpha]$.

For every $u \in T$*, we let* $\mathrm{ord}(u)$, $\mathrm{fun}_1(u)$, $\mathrm{fun}_2(u)$, $\mathrm{fun}_3(u)$*, and* $\mathrm{cst}(u)$ *be elements such that* $u \in T[\mathrm{ord}(u)]$ *and* $u = \langle \mathrm{fun}_1(u), \mathrm{fun}_2(u), \mathrm{fun}_3(u), \mathrm{cst}(u)\rangle$. *Furthermore, for every* $u \in T$*, we let* $u{\restriction}\beta$ *denote* $\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$ *when* $\beta = 0$*, and when* $\beta > 0$*,*

$$u{\restriction}\beta = \langle \mathrm{fun}_1(u){\restriction}\beth_\beta, \mathrm{fun}_2(u){\restriction}\beth_\beta, \mathrm{fun}_3(u){\restriction}\beth_\beta, \mathrm{cst}(u)\cap\beta\rangle.$$

The elements $u, v \in T$ form a tree when they are ordered by

$$u \lhd v \textit{ iff } u = v{\restriction}\mathrm{ord}(u) \textit{ and } \mathrm{ord}(u) \in \mathrm{cst}(v).$$

The notation $u \unlhd v$ stands for $u \lhd v$ or $u = v$. For a $\lhd$-increasing chain $\langle u_i \mid i < \theta\rangle$ of elements in T, $\theta < \kappa$, we write $\bigcup_{i<\theta} u_i$ for the following element in T:

$$\Big\langle \bigcup_{i<\theta} \mathrm{fun}_1(u_i), \bigcup_{i<\theta} \mathrm{fun}_2(u_i), \bigcup_{i<\theta} \mathrm{fun}_3(u_i), C\Big\rangle,$$

where C is the closure of $\bigcup_{i<\theta} \mathrm{cst}(u_i)$.

Definition 3.2 *For every $s, t, r \in {}^{\kappa}2$ define*

$$C_{s,t,r} = \{0\} \cup \{\delta < \kappa \mid \langle V_\delta, \in, P \cap V_\delta, s{\restriction}\delta, t{\restriction}\delta, r{\restriction}\delta\rangle \prec \langle V_\kappa, \in, P, s, t, r\rangle \models \phi\}.$$

(Then for all nonzero $\delta \in C_{s,t,r}$, $\beth_\delta = \delta$.) We let T^1 be the set of all $u \in T$ such that for some $s, t, r \in {}^{\kappa}2$, the following conditions are satisfied:

> $\mathrm{fun}_1(u) \subseteq s$, $\mathrm{fun}_2(u) \subseteq t$, *and* $\mathrm{fun}_3(u) \subseteq r$*;*
>
> $\mathrm{ord}(u) \in C_{s,t,r}$ *and* $\mathrm{cst}(u) = C_{s,t,r} \cap \mathrm{ord}(u)$*.*

Definition 3.3 *For each $\alpha < \kappa$ define a lexicographic order $\lessdot_\alpha$ as follows: for all elements $\eta, \nu \in {}^{(\beth_\alpha)}2$,*

$$\eta \lessdot_\alpha \nu \textit{ iff } \eta \neq \nu \textit{ and } \eta(\xi) < \nu(\xi), \textit{for } \xi = \min\{\zeta < \beth_\alpha \mid \eta(\zeta) \neq \nu(\zeta)\}.$$

Define

$$T[\lessdot] = \{u \in T \mid \mathrm{fun}_1(u) \lessdot_{\mathrm{ord}(u)} \mathrm{fun}_2(u)\}.$$

For every $u \in T$, denote the tuple $\langle \mathrm{fun}_2(u), \mathrm{fun}_1(u), \mathrm{fun}_3(u), \mathrm{cst}(u)\rangle$ by u^{-1} (the order of the first and the second elements are exchanged).

Remark. For every $u \in T^1$, $u^{-1} \in T^1$ by the assumption (1).

Definition 3.4 *Define Suc^+ to be the set $\{\beta+1 \mid \beta$ is a successor ordinal $\}$. Assume π is a surjective function from $\{0\} \cup \mathrm{Suc}^+$ onto $\{\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle\} \cup T[\lessdot]$ such that*

if $\pi(\alpha) = u$ *then either* $\alpha = 0$ *and* $u = \langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$*, or else* $\mathrm{ord}(u) < \alpha$*;*

for every $u \in \{\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle\} \cup T[\lessdot]$*, the set* $\{\alpha \in \mathrm{Suc}^+ \mid \pi(\alpha) = u\}$ *is unbounded in* κ*.*

We define T^2_λ*, for fixed regular* $\lambda < \kappa$*, to be the smallest subset of* T *satisfying the following conditions.*

1) $\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$ *is in* T^2_λ*.*
2) *If* $u \in T^2_\lambda$ *then* $u^{-1} \in T^2_\lambda$*.*
3) T^2_λ *contains every* $u \in T[\lessdot]$ *having the properties:*
    i) *If* $\sup \mathrm{cst}(u) < \mathrm{ord}(u)$ *then* $\mathrm{ord}(u) \in \mathrm{Suc}^+$ *and for the maximal element* $\gamma \in \mathrm{cst}(u)$*,* $\pi\big(\mathrm{ord}(u)\big) = u{\restriction}\gamma \in T^1 \cup T^2_\lambda$*;*
    ii) *if* $\sup \mathrm{cst}(u) = \mathrm{ord}(u)$*, then* $\mathrm{cst}(u) \cap \mathrm{Suc}^+$ *is nonempty,* $\mathrm{cf}(\mathrm{ord}(u)) < \lambda$*, and for every* $\beta \in \mathrm{cst}(u)$*,* $u{\restriction}\beta \in T^1 \cup T^2_\lambda$*.*

We shall need only a restricted part of T, so we change the notation a little bit.

Definition 3.5 *For each* $\alpha < \kappa$*, let* $\mathrm{Con}\big({}^{(\beth_\alpha)}2\big)$ *be the family of functions* η *satisfying that* $\eta \in {}^{(\beth_\alpha)}2$ *for some* $\alpha < \kappa$*,* $\eta(\xi) = 0$ *if* $\xi < \beth_0$*, and for all* $\beta < \alpha$ *and* $\xi \in \beth_{\beta+1} \smallsetminus \beth_\beta$*,* $\eta(\xi) = \eta(\beth_\beta)$*. Define the restricted part of* T *to be*

$$U = \Big\{u \in T^1 \cup T^2_\lambda \mid u = \langle\emptyset,\emptyset,\emptyset,\emptyset\rangle \text{ or } \\ \mathrm{fun}_1(u), \mathrm{fun}_2(u), \mathrm{fun}_3(u) \in \mathrm{Con}\big({}^{(\beth_{\mathrm{ord}(u)})}2\big)\Big\}.$$

Denote $T^1 \cap U$ *by* U^1 *and* $T^2_\lambda \cap U$ *by* U^2_λ*. We write that* $u \in U$ *is a successor of* v *when* $v \in U$ *and there is no* $w \in U$ *with* $v \lhd w \lhd u$*. An element* $u \in U$ *is called a successor if* $u = \langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$ *or there is* $v \in U$ *such that* u *is a successor of* v*. If* u *is not a successor,* u *is called a limit. For all* $\alpha < \kappa$*,* $U[\alpha]$ *denote* $T[\alpha] \cap U$ *and* $U[<\alpha]$ *stand for* $T[<\alpha] \cap U$*. When* $u \in U$*, we write that "for all* $v \lhd u$*" when we mean that "for all* $v \in U$ *with* $v \lhd u$*".*

Remark. By the assumption (2) at the beginning of this section and the use of elementary submodels in Definition 3.2, our restriction of ${}^{(\beth_\alpha)}2$ to $\mathrm{Con}\big({}^{(\beth_\alpha)}2\big)$ is harmless. However this restriction turned out to be useful in

the forthcoming Definition 3.9 and the property Lemma 3.16(e) (we want that for successor $u \in U$ with c^u increasing, the information $\mathrm{end}(p_u)$ is determined by a single point $\zeta \in \mathrm{ran}(c^u)$ alone, see the definitions 3.9 and 3.11 below for unexplained notation).

Note also that an element $u \in U^1$ is a successor of $v \in U$ only if $v \in U^1$. However, for every $v \in U^1$ there is $u \in U^2_\lambda$ which is a successor of v. When $u \in U^1$ is a limit point then it is a limit of elements in U^1. Besides, $u \in U^2_\lambda$ is a limit point only if it is a limit of elements in U^2_λ (see the proof of Fact 3.6(b) below).

Fact 3.6

a) *For all $\triangleleft$-increasing chain $\langle u_i \mid i < \theta \rangle$ of elements in U^2_λ, $\mathrm{cf}(\theta) < \lambda$ and the tuple $\bigcup_{i<\theta} u_i$ is in U^2_λ.*

b) *For all $\triangleleft$-increasing chain $\bar{u} = \langle u_i \mid i < \theta \rangle$ of elements in U^1, if $\bar{u}$ has some upper bound in U (with respect to the order $\triangleleft$) then the tuple $\bigcup_{i<\theta} u_i$ is in U^1.*

Proof. a) This property is an obvious consequence of Definition 3.4(3.ii).

b) Suppose that $w \in U$ is an upper bound for $\bar{u}$. Let v be the smallest element in Uwith $v \trianglelefteq w$ and $u_i \triangleleft v$ for every $i < \theta$. Then v is a limit of the elements $u_i \in U^1$, $i < \theta$.

Suppose v is in U^2_λ. By Definition 3.4(3.i), $\mathrm{cst}(v)$ does not contain a maximal element, $\mathrm{cst}(v) = \bigcup_{i<\theta} \mathrm{cst}(u_i)$ since $\bar{u}$ is $\triangleleft$-increasing, and for every $\beta \in \mathrm{cst}(u)$, $u{\restriction}\beta \in U$. Since Suc^+ contains only successor ordinals and each u_i is in U^1, $\mathrm{cst}(u_i) \cap \mathrm{Suc}^+ = \emptyset$ for every $i < \theta$. Hence $\mathrm{cst}(v)$ is disjoint from Suc^+ contrary to Definition 3.4(3.ii).

It follows that v must be in U^1 and there are $s, t, r \in {}^\kappa 2$ such that

$$\begin{aligned}&\mathrm{fun}_1(v) \subseteq s, \mathrm{fun}_2(v) \subseteq t, \mathrm{fun}_3(v) \subseteq r,\\ &\mathrm{ord}(v) \in C_{s,t,r}, \text{and } \mathrm{cst}(v) = C_{s,t,r} \cap \mathrm{ord}(v).\end{aligned}$$

For every $i < \theta$ and $\alpha_i = \mathrm{ord}(u_i)$, $u_i \triangleleft v$ implies that $\alpha_i \in \mathrm{cst}(v) \subseteq C_{s,t,r}$. Hence, for each $i < \theta$,

$$\langle V_{\alpha_i}, \in, P \cap V_{\alpha_i}, s{\restriction}\alpha_i, t{\restriction}\alpha_i, r{\restriction}\alpha_i \rangle \prec \langle V_\kappa, \in, P, s, t, r \rangle \models \phi,$$

and for $\delta = \bigcup_{i<\theta} \alpha_i$,

$$\langle V_\delta, \in, P \cap V_\delta, s{\restriction}\delta, t{\restriction}\delta, r{\restriction}\delta \rangle \prec \langle V_\kappa, \in, P, s, t, r \rangle \models \phi.$$

Consequently, the tuple $\langle s\restriction\delta, t\restriction\delta, r\restriction\delta, C_{s,t,r}\cap\delta\rangle = \bigcup_{i<\theta} u_i$ is in U^1 (note that $\beth_{\alpha_i} = \alpha_i$, $\beth_\delta = \delta$, and by the choice of v, $v = \bigcup_{i<\theta} u_i$). $\blacksquare_{3.6}$

Definition 3.7 *For every $\beta < \kappa$ and $\gamma < \beth_{\beta+1}$ we define a function c^β_γ with domain $\beth_\beta$ as follows: for all $\xi < \beth_\beta$,*

$$c^\beta_\gamma(\xi) = \big(\beth_\beta \cdot (\gamma+1)\big) + \xi,$$

where $\cdot$ and $+$ are the ordinal multiplication and addition respectively. Write $\mathcal{E}$ for the family of functions $\{f\restriction A \mid f \in E \text{ and } A \subseteq \mathrm{dom}(f)\}$, where E is the set $\bigcup\big\{\{c^\beta_\gamma, (c^\beta_\gamma)^{-1}\} \mid \beta < \kappa \text{ and } \gamma < \beth_{\beta+1}\big\}$. The reflection point of $d \in \mathcal{E} \smallsetminus \{\emptyset\}$, denoted be $\mathrm{ref}(d)$, is the unique ordinal β for which there is $\gamma < \beth_{\beta+1}$ satisfying that either $d \subseteq c^\beta_\gamma$ or $d \subseteq \big(c^\beta_\gamma\big)^{-1}$.

Fact 3.8

a) *For all increasing $d, e \in \mathcal{E}$ and $Y = \mathrm{ran}(d) \cap \mathrm{ran}(e)$, $d^{-1}\restriction Y = e^{-1}\restriction Y$.*

b) *For every $e \in \mathcal{E}$, either e is increasing and all the elements in $\mathrm{ran}(e)$ have the same cardinality, or otherwise, e is decreasing and all the elements in $\mathrm{dom}(e)$ have the same cardinality.*

c) *For all $e \in \mathcal{E}$ and $\xi < \zeta \in \mathrm{dom}(e)$, $\zeta - \xi = e(\zeta) - e(\xi)$.*

Definition 3.9 *First we need some auxiliary means used in this definition only. For all functions p and e, $p \uplus e$ is the function $p \cup \big(e\restriction(\mathrm{dom}(e) \smallsetminus \mathrm{dom}(p))\big)$. Let Ref^1 be the set of all limit ordinals below κ and Ref^2 be the set of all successor ordinals below κ. For every $\alpha < \kappa$ let $\ll_\alpha$ be a fixed well-ordering of $U[\alpha]$ and define a well-ordering of Uby*

$$u \ll v \text{ iff } \mathrm{ord}(u) < \mathrm{ord}(v) \text{ or } (\mathrm{ord}(u) = \mathrm{ord}(v) = \alpha \text{ and } u \ll_\alpha v).$$

For each $w \in U$, denote the set $\{w' \mid w' \ll w\}$ by $U[\ll w]$. Furthermore, let $\mathrm{id}(u)$, for $u \in U$, denote the identity function

$$\{\langle\xi,\xi\rangle \mid \xi < \mathrm{ord}(u) \text{ and } \mathrm{fun}_1(u)\restriction\xi+1 = \mathrm{fun}_2(u)\restriction\xi+1\}.$$

Now define for each $u \in U$ a function p_u as follows.

a) *First of all $p_u = \emptyset$ for $u = \langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$.*

b) *Suppose* $u \in T[\lessdot]$, *u is a successor of* v, *and for* v *the function* p_v *is already defined. Fix an ordinal* $\beta < \mathrm{ord}(u)$ *as follows:*

i) *Suppose* $u \in U^1$. *For all* $w \in U$, *define inductively that*

$$\beta'_w = \min\Big(\mathrm{Ref}^1 \smallsetminus \big(\mathrm{dom}(\mathrm{id}(u)) \cup (\mathrm{ord}(w)+1) \cup \{\beta'_{w'} \mid w' \in U[\ll w]\}\big)\Big).$$

Fix β *to be* β'_v.

ii) *Suppose* $u \in U^2_\lambda$. *By Definition 3.4,* $\mathrm{ord}(u) \in \mathrm{Suc}^+$ *and there is a unique* $\beta \in \mathrm{Ref}^2$ *with* $\mathrm{ord}(u) = \beta + 1$. *(Note that* $v = \pi(\mathrm{ord}(u))$.*)*

Assume $\langle w^\beta_{\gamma'} \mid \gamma' < \theta \rangle$, *for* $\theta \leq \beth_{\beta+1}$, *is a fixed enumeration of* $T[\beta]$ *without repetition. Let* γ *be the ordinal for which* $u{\restriction}\beta = w^\beta_\gamma$ *holds. We define*

$$p_u = \begin{cases} \mathrm{id}(u) \uplus (p_v \uplus c^\beta_\gamma) & \textit{if } \mathrm{ran}(p_v) \textit{ is ordinal;} \\ \mathrm{id}(u) \uplus \big(p_v \uplus (c^\beta_\gamma)^{-1}\big) & \textit{otherwise.} \end{cases}$$

c) *Suppose* $u \in T[\lessdot]$, *u is a limit, and for all* $v \lhd u$, *functions* p_v *are defined. Then define* p_u *to be* $\bigcup_{v \lhd u} p_v$. *(By the definition of* $\uplus$, $p_w \subseteq p_v$ *for all* $w \lhd v \lhd u$.*)*

d) *For all* $u \in U \smallsetminus T[\lessdot]$ *define* p_u *to be* $(p_{u^{-1}})^{-1}$.

For every successor $u \in U \smallsetminus \{\langle \emptyset, \emptyset, \emptyset, \emptyset \rangle\}$, *say a successor of* $v \in U$, *there is unique* $e \in \mathcal{E}$ *such that either* $e = \emptyset$, *or otherwise,* $\mathrm{dom}(e) = \mathrm{dom}(p_u) \smallsetminus \mathrm{dom}(p_v)$ *and* $p_u = p_v \cup e$. *We denote this* e *by* c^u.

Remark. The part $\mathrm{id}(u)$ in the definition above is needed first time in Lemma 3.20(d) to ensure that all the functions have arbitrary large extensions. Note also that p_u might be $\mathrm{id}(u) = \{\langle \xi, \xi \rangle \mid \xi < \beth_\beta\}$ when $\mathrm{ord}(u) = \beta + 1$, $u \in U^2_\lambda$ is a successor of $\langle \emptyset, \emptyset, \emptyset, \emptyset \rangle$, and $\mathrm{fun}_1(u){\restriction}\beth_\beta = \mathrm{fun}_2(u){\restriction}\beth_\beta$.

Fact 3.10

a) *If* $u \in U$, $\xi \in \mathrm{dom}(p_u)$, *and* $p_u(\xi) = \xi$, *then* $p_u(\zeta) = \zeta$ *for all* $\zeta \leq \xi$.

b) *For every* $u \in U$, p_u *is a partial function from* $\beth_{\mathrm{ord}(u)}$ *into* $\beth_{\mathrm{ord}(u)}$.

c) *For all* $u, v \in U$, $u \lhd v$ *implies* $p_u \subsetneq p_v$.

d) *For every successor* $u \in U \setminus \{\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle\}$, $\mathrm{dom}(p_u)$ *is the cardinal* $\beth_{\mathrm{ref}(c^u)}$ *iff* c^u *is* $\emptyset$ *or* c^u *is increasing, and* $\mathrm{ran}(p_u)$ *is the cardinal* $\beth_{\mathrm{ref}(c^u)}$ *iff* c^u *is* $\emptyset$ *or* c^u *is decreasing.*

e) *For all limit points* $u \in U$, $\mathrm{dom}(p_u) = \bigcup_{v \lhd u} \mathrm{dom}(p_v) = \mathrm{ran}(p_u) = \bigcup_{v \lhd u} \mathrm{ran}(p_v) = \beth_{\mathrm{ord}(u)}$.

f) *For all limit points* $u \in U^2_\lambda$, $\mathrm{dom}(p_u)$ *is a cardinal of cofinality less than* λ.

g) *Suppose that both* u *and* v *are successor elements in* U. *If* $c^u \cap c^v \neq \emptyset$ *then* u *and* v *are successors of the same element,* $c^u = c^v$, *and for* $\beta = \mathrm{ref}(c^u) = \mathrm{ref}(c^v)$, $u{\restriction}\beta = v{\restriction}\beta$.

Proof. The proofs of (b)–(e) are straightforward inductions on the tree order $\lhd$. Note that for every limit $u \in U$, $u = \bigcup_{v \lhd u} v$ by Fact 3.6. Note also that in Definition 3.9(b.i), when $u \in U^1$ is a successor of v, the following holds:

$$\beta'_v < \mathrm{card}(U[\ll v])^+ \leq \beth_{\mathrm{ord}(v)+1}{}^+ < \mathrm{ord}(u),$$

since $\mathrm{card}(U[\ll v]) \leq \beth_{\mathrm{ord}(v)+1}$, $\mathrm{ord}(u) = \beth_{\mathrm{ord}(u)}$, and Ref^1 is the set of all limit ordinals.

f) By (e) $\mathrm{dom}(p_u)$ is the cardinal $\beth_{\mathrm{ord}(u)}$. By Definition 3.4(3.ii), $\mathrm{cf}(\mathrm{ord}(u)) < \lambda$.

g) Let w^1 and w^2 be such that u is a successor of w^1 and v is a successor of w^2. By Definition 3.9(b), $p_u = p_{w^1} \cup c^u$ and $p_v = p_{w^2} \cup c^v$. If $w_1 = w_2$ and $c^u \cap c^v \neq \emptyset$ then there are $\beta < \min\{\mathrm{ord}(u), \mathrm{ord}(v)\}$ and $\gamma < \beth_{\beta+1}$ such that $c^u = c^v \subseteq c^\beta_\gamma$ and $u{\restriction}\beta = w^\beta_\gamma = v{\restriction}\beta$, where w^β_γ is given in Definition 3.9(b).

Suppose $u \in U^1$. If $v \in U^1$ and $w^1 \neq w^2$ then $\mathrm{ref}(c^u) \neq \mathrm{ref}(c^v)$ because the mapping $w \mapsto \beta'_w$, given in Definition 3.4, is injective. Hence $c^u \cap c^v = \emptyset$. Assume $v \in U^2_\lambda$. Then $\mathrm{ref}(c^v) \in \mathrm{Ref}^2 \setminus \mathrm{Ref}^1$, and because $\mathrm{ref}(c^u) \in \mathrm{Ref}^1$, $c^u \cap c^v = \emptyset$ holds. Similarly, $c^u \cap c^v = \emptyset$ if $u \in U^2_\lambda$ and $v \in U^1$.

Suppose both $u \in U^2_\lambda$ and $v \in U^2_\lambda$. Then the equations $\pi(\mathrm{ord}(u)) = w^1$ and $\pi(\mathrm{ord}(v)) = w^2$ hold, and there are $\beta^1, \beta^2 \in \mathrm{Ref}^2$ with $\mathrm{ord}(u) = \beta^1 + 1$ and $\mathrm{ord}(v) = \beta^2 + 1$. If $\mathrm{ord}(u) \neq \mathrm{ord}(v)$ then $\mathrm{ref}(c^u) = \beta^1 \neq \beta^2 = \mathrm{ref}(c^v)$ and hence $c^u \cap c^v = \emptyset$. On the other hand, when $\mathrm{ord}(u)$ equals $\mathrm{ord}(v)$, $w^1 = \pi(\mathrm{ord}(u)) = \pi(\mathrm{ord}(v)) = w^2$. ■

Definition 3.11 *We define* Seq *to be the set of all pairs* $\langle \bar{u}, W\rangle$ *satisfying the following conditions:*

a) W *is nonempty.*

b) *For some* $n < \omega$, $\bar{u}$ *is a sequence* $\langle u_i \mid i < n\rangle$ *of elements in* $U \smallsetminus \{\langle \emptyset, \emptyset, \emptyset, \emptyset\rangle\}$.

c) *Let* W_0 *be* W*. Inductively for every* $i < n-1$, $W_i \subseteq \mathrm{dom}(p_{u_i})$ *and* $W_{i+1} = p_{u_i}[W_i]$.

d) *For every* $i < n-1$, $\mathrm{fun}_2(u_i){\restriction}\sup^+(W_{i+1}) = \mathrm{fun}_1(u_{i+1}){\restriction}\sup^+(W_{i+1})$.

For every $\langle \bar{u}, W\rangle$ *in* Seq *there is the natural sequence* $\bar{g}^{\bar{u},W}$ *of functions defined as follows:*

$$\bar{g}^{\bar{u},W} = \langle g_i^{\bar{u},W} \mid i < \mathrm{lh}(\bar{u})\rangle,$$

where each $g_i^{\bar{u},W}$ *is a shorthand for* $p_{u_i}{\restriction}W_i$*. The composition* $g_{\mathrm{lh}(\bar{u})-1}^{\bar{u},W} \circ \ldots \circ g_0^{\bar{u},W}$ *is denoted by* $g^{\bar{u},W}$*. For all sequences* $\bar{f} = \langle f_i \mid i < n\rangle$, $1 \leq n < \omega$, *which are of the form* $\bar{g}^{\bar{u},W}$, *for some fixed* $\langle \bar{u}, W\rangle \in \mathrm{Seq}$ *with* $\mathrm{lh}(\bar{u}) = n$, *we shall use the following notation:*

for each $i < \mathrm{lh}(\bar{u})$,

$$\begin{aligned}\mathrm{ind}(f_i) &= u_i,\\ \mathrm{beg}(f_i) &= \mathrm{fun}_1(u_i){\restriction}\sup^+(\mathrm{dom}(f_i)),\\ \mathrm{end}(f_i) &= \mathrm{fun}_2(u_i){\restriction}\sup^+(\mathrm{ran}(f_i)).\end{aligned}$$

$\mathrm{beg}(f) = \mathrm{beg}(f_0)$ *and* $\mathrm{end}(f) = \mathrm{end}(f_{\mathrm{lh}(\bar{u})-1})$*;*

f *is the composition* $f_{\mathrm{lh}(\bar{u})-1} \circ \ldots \circ f_0$*;*

for $i < \mathrm{lh}(\bar{u})$, $f_{\leq i}$ *is a shorthand for* $f_i \circ \ldots \circ f_0$, *and for all* $\xi \in \mathrm{dom}(f)$,

$$f_{<i}(\xi) = \begin{cases} \xi & \textit{if } i = 0;\\ f_{i-1} \circ \ldots \circ f_0(\xi) & \textit{otherwise.}\end{cases}$$

Definition 3.12 *A sequence* $\bar{f} = \bar{g}^{\bar{u},W}$, $\langle \bar{u}, W\rangle \in \mathrm{Seq}$, *is called minimal if the following two conditions are satisfied:*

a) *for all* $i < \mathrm{lh}(\bar{f})$ *and* $v \triangleleft \mathrm{ind}(f_i)$*,* $\mathrm{dom}(f_i) \not\subseteq \mathrm{dom}(p_v)$*;*

b) *there are no indices* $i \leq j < \mathrm{lh}(\bar{f})$ *satisfying that the composition* $f_j \circ \ldots \circ f_i$ *is identity and* $\mathrm{beg}(f_i) = \mathrm{end}(f_j)$*.*

Let $\bar{\mathcal{F}}$ *be the set* $\{\bar{g}^{\bar{u},W} \mid \langle \bar{u}, W\rangle \in \mathrm{Seq}$ *and* $\bar{g}^{\bar{u},W}$ *is minimal*$\}$*. We abbreviate* $\{f \mid \bar{f} \in \bar{\mathcal{F}}$ *and* $\mathrm{lh}(\bar{f}) = 1\}$ *to* $\mathcal{F}_1$*.*

Fact 3.13

a) *For all* $\langle \bar{u}, W\rangle \in \mathrm{Seq}$*, either* $g^{\bar{u},W}$ *is the identity function and* $\mathrm{beg}(g^{\bar{u},W}) = \mathrm{end}(g^{\bar{u},W})$*, or otherwise, there is a sequence* $\bar{f} \in \bar{\mathcal{F}}$ *such that* $\mathrm{lh}(\bar{f}) \leq \mathrm{lh}(\bar{u})$*,* $f = g^{\bar{u},W}$*,* $\mathrm{beg}(f) = \mathrm{beg}(g^{\bar{u},W})$*, and* $\mathrm{end}(f) = \mathrm{end}(g^{\bar{u},W})$*.*

b) *For every* $q \in \mathcal{F}_1$ *there is* $\theta \in \mathrm{dom}(q)$ *such that for all* $\xi \in \mathrm{dom}(q) \setminus \theta$*,* $q(\xi) \neq \xi$*.*

c) *For every* $q \in \mathcal{F}_1$ *with* $\mathrm{ind}(q) = u$ *a successor,* c^u *is nonempty.*

Note that for all functions x and sets X, $x \restriction X$ means the restricted function $x \restriction (\mathrm{dom}(x) \cap X)$, so we do not demand that $X \subseteq \mathrm{dom}(x)$ in any such restrictions.

Lemma 3.14 *For all nonempty* $q \in \mathcal{F}_1$*,*

$$\mathrm{ind}(q) \text{ is a successor if, and only if, } \sup^+(\mathrm{dom}(q)) \neq \sup^+(\mathrm{ran}(q)).$$

Moreover, if $\mathrm{ind}(q) = u$ *is a limit, then*

$$\sup^+(\mathrm{dom}(q)) = \sup^+(\mathrm{ran}(q)) = \mathrm{dom}(p_u) = \mathrm{ran}(p_u) = \beth_{\mathrm{ord}(u)}.$$

Proof. First of all recall, that q is not identity, Fact 3.13(b). Suppose first that $\mathrm{ind}(q) = u$ is a successor of $v \in U$. Then $q \subseteq p_v \cup e$ for $e = c^u \restriction \mathrm{dom}(q)$. Abbreviate $\mathrm{ref}(e)$ by γ. We have $\gamma \geq \mathrm{ord}(v)$ and $\mathrm{dom}(p_v) \cup \mathrm{ran}(p_v) \subseteq \beth_{\mathrm{ord}(v)} \leq \beth_\gamma$. Because e is nonempty, the claim follows from the facts that $\mathrm{dom}(e) \cap \beth_\gamma \neq \emptyset$ implies $\mathrm{dom}(e) \subseteq \beth_\gamma$ and $\mathrm{ran}(e) \cap \beth_\gamma = \emptyset$, and on the other hand, $\mathrm{dom}(e) \cap \beth_\gamma = \emptyset$ implies $\mathrm{ran}(e) \subseteq \beth_\gamma$.

Assume $\mathrm{ind}(q)$ is a limit. Denote $\mathrm{ind}(q)$ by u and $\beth_{\mathrm{ord}(u)}$ by μ. Because $\mathrm{dom}(p_u) = \mathrm{ran}(p_u) = \mu$ it suffices to prove that $\sup^+(\mathrm{dom}(q)) \geq \mu$ and $\sup^+(\mathrm{ran}(q)) \geq \mu$.

Let $\theta < \kappa$ be such that $\langle v_i \mid i < \theta\rangle$ is a $\lhd$-increasing enumeration of the elements $w \lhd u$. We know that for all ordinals i in $\{j + (2n+1) \mid j < \theta$ is a limit ordinal or 0, and $n < \omega\}$, $\mathrm{dom}(p_{v_i})$ is a cardinal. If $\sup^+(\mathrm{dom}(q)) < \mu$ then there would be $i < \theta$ with $\mathrm{dom}(q) \subseteq \mathrm{dom}(p_{v_i})$ contrary to Definition 3.12(a). So $\mathrm{dom}(q)$ must be unbounded in μ.

Besides, we know that for all ordinals i in the set $I = \{j + 2n \mid j < \theta$ is a limit ordinal and $n < \omega\}$, $\mathrm{ran}(p_{v_i})$ is a cardinal. So if $\sup^+(\mathrm{ran}(q)) < \mu$ and $i \in I$ is such that $\mathrm{ran}(q) \subseteq \mathrm{ran}(p_{v_i})$, then $\mathrm{dom}(q) \subseteq \mathrm{dom}(p_{v_i})$ since q is injective and $\mathrm{ran}(p_{v_i})$ is an ordinal. Thus $\mathrm{ran}(q)$ is also unbounded in μ.

$\blacksquare_{3.14}$

Lemma 3.15

a) *Suppose $p, q \in \mathcal{F}_1$ and $e \in \mathcal{E}$ is a nonempty increasing function with $e \subseteq p \cap q$ and* $\mathrm{ref}(e) = \beta$. *Then for $X = \mathrm{dom}(p) \cap \mathrm{dom}(q) \cap \beth_\beta$, $p{\restriction}X = q{\restriction}X$ and* $\mathrm{ind}(p){\restriction}\beta = \mathrm{ind}(q){\restriction}\beta$. *Particularly,* $\mathrm{beg}(p){\restriction}\sup^+(X) = \mathrm{beg}(q){\restriction}\sup^+(X)$.

b) *For all $q \in \mathcal{F}_1$, if* $\sup^+(\mathrm{dom}(q))$ *is a cardinal, then* $\sup^+(\mathrm{dom}(q)) \leq \sup^+(\mathrm{ran}(q))$.

c) *If $\bar{f} \in \bar{\mathcal{F}}$ and* $\sup^+(\mathrm{dom}(f_0)) < \sup^+(\mathrm{ran}(f_0))$ *then* $\sup^+(\mathrm{dom}(f_i)) < \sup^+(\mathrm{ran}(f_i))$ *for every $i < \mathrm{lh}(\bar{f})$.*

d) *For all $\bar{f} \in \bar{\mathcal{F}}$ and $i < \mathrm{lh}(\bar{f})$,* $\sup^+(\mathrm{dom}(f_i) \cup \mathrm{ran}(f_i)) \leq \sup^+(\mathrm{dom}(f) \cup \mathrm{ran}(f))$.

e) *For all $\bar{f} \in \bar{\mathcal{F}}$, if* $\sup^+(\mathrm{ran}(f))$ *is a cardinal μ and* $\mathrm{dom}(f) \subseteq \mu$ *then* $\sup^+(\mathrm{dom}(f)) = \mu$. *Furthermore, if* $\mathrm{dom}(f) = \mu$ *then* $\mathrm{dom}(f_i) = \mathrm{ran}(f_i) = \mu$ *for each $i < \mathrm{lh}(\bar{f})$.*

Proof. a) By Definition 3.9, there are successors $u, v \in U$ with $u \trianglelefteq \mathrm{ind}(p)$, $v \trianglelefteq \mathrm{ind}(q)$, $c^u{\restriction}\mathrm{dom}(e) = p_u{\restriction}\mathrm{dom}(e) = e = p_v{\restriction}\mathrm{dom}(e) = c^v{\restriction}\mathrm{dom}(e)$, and $\mathrm{dom}(p_u) = \mathrm{dom}(p_v) = \beth_\beta$. By Fact 3.10(g), u and v are successors of the same element, say $w \in U$, $c^u = c^v$, and $u{\restriction}\beta = v{\restriction}\beta$. Therefore we have

$$p{\restriction}\beth_\beta \subseteq p_u = p_w \cup c^u = p_w \cup c^v = p_v \supseteq q{\restriction}\beth_\beta,$$

and

$$\mathrm{beg}(p){\restriction}\beth_\beta \subseteq \mathrm{fun}_1(u){\restriction}\beth_\beta = \mathrm{fun}_1(v){\restriction}\beth_\beta \supseteq \mathrm{beg}(q){\restriction}\beth_\beta.$$

b) Suppose that $\sup^+(\mathrm{dom}(q)) > \sup^+(\mathrm{ran}(q))$. Then by Lemma 3.14, $\mathrm{ind}(q)$ must be a successor. Denote $c^{\mathrm{ind}(q)}\restriction\mathrm{dom}(q)$ by d. Necessarily d is decreasing and $\mathrm{dom}(d)$ is an end segment of $\mathrm{dom}(q)$. By Fact 3.8(b), $\mathrm{card}\big(\sup^+(\mathrm{dom}(d))\big) = \mathrm{card}\big(\min\mathrm{dom}(d)\big)$. Thus $\sup^+(\mathrm{dom}(q)) = \sup^+(\mathrm{dom}(d)) > \min\mathrm{dom}(d)$ is not a cardinal.

c) Suppose, contrary to the claim, that there is $j \in \{1, \dots, \mathrm{lh}(\bar{f}) - 1\}$ with $\sup^+(\mathrm{dom}(f_j)) \geq \sup^+(\mathrm{ran}(f_j))$. We may assume that j is the smallest possible index with this property.

Suppose first that $\sup^+(\mathrm{dom}(f_j)) = \sup^+(\mathrm{ran}(f_j))$. Then for $u = \mathrm{ind}(f_j)$, $\sup^+(\mathrm{dom}(f_j))$ is the cardinal $\beth_{\mathrm{ord}(u)}$ by Lemma 3.14. It follows from the equation $\mathrm{dom}(f_j) = \mathrm{ran}(f_{j-1}) = \mathrm{dom}({f_{j-1}}^{-1})$ and by applying (b) to ${f_{j-1}}^{-1}$, that $\sup^+(\mathrm{dom}({f_{j-1}}^{-1})) \leq \sup^+(\mathrm{ran}({f_{j-1}}^{-1}))$. However, then $\sup^+(\mathrm{dom}(f_{j-1})) \geq \sup^+(\mathrm{ran}(f_{j-1}))$, contrary to the choice of j.

So suppose $\sup^+(\mathrm{dom}(f_j)) > \sup^+(\mathrm{ran}(f_j))$. Note that $\sup^+(\mathrm{dom}(f_{j-1})) < \sup^+(\mathrm{ran}(f_{j-1}))$. Abbreviate $\mathrm{ind}(f_{j-1})$ by u and $\mathrm{ind}(f_j)$ by v. By Lemma 3.14, there are $w^1, w^2 \in U$ such that u is a successor of w^1 and v is a successor of w^2. Denote $\mathrm{ref}(c^u)$ by γ^1, $\mathrm{ref}(c^v)$ by γ^2, $c^u\restriction\mathrm{dom}(f_{j-1})$ by d^1, and $c^v\restriction\mathrm{dom}(f_j)$ by d^2. Then d^1, d^2 are nonempty, d^1 is increasing, d^2 is decreasing, and

$$\begin{aligned}
&f_{j-1} \subseteq p_u = p_{w^1} \cup d^1,\\
&\mathrm{ran}(p_{w^1}) \subseteq \beth_{\gamma^1},\\
&\mathrm{ran}(d^1) \subseteq \beth_{\gamma^1+1} \setminus \beth_{\gamma^1},\\
&f_j \subseteq p_v = p_{w^2} \cup d^2\\
&\mathrm{dom}(p_{w^2}) \subseteq \beth_{\gamma^2},\\
&\mathrm{dom}(d^2) \subseteq \beth_{\gamma^2+1} \setminus \beth_{\gamma^2}.
\end{aligned}$$

Because $\mathrm{ran}(f_{j-1}) = \mathrm{dom}(f_j)$, it follows that $\gamma_1 = \gamma_2$ and $\mathrm{ran}(d^1) = \mathrm{dom}(d^2)$. By Fact 3.8(a), $d^1 = (d^2)^{-1}$, and hence $c^u \cap c^{(v^{-1})} \neq \emptyset$ (the notation v^{-1} is explained in Definition 3.3). By Fact 3.10(g), $w^1 = (w^2)^{-1}$ and $u\restriction\gamma_1 = v^{-1}\restriction\gamma_1$. Consequently, $f_{j-1} = {f_j}^{-1}$ and for $\theta = \sup^+(\mathrm{dom}(f_{j-1})) = \sup^+(\mathrm{ran}(f_j)) \leq \beth_{\gamma_1}$, $\mathrm{beg}(f_{j-1}) = \mathrm{fun}_1(u)\restriction\theta = \mathrm{fun}_1(v^{-1})\restriction\theta = \mathrm{fun}_2(v)\restriction\theta = \mathrm{end}(f_j)$ contrary to the minimality of $\bar{f}$.

d) Denote $\sup^+(\mathrm{dom}(f)\cup\mathrm{ran}(f))$ by θ. To reach a contradiction let $j < \mathrm{lh}(\bar{f})$ be the smallest index with $\sup^+(\mathrm{dom}(f_j)\cup\mathrm{ran}(f_j)) > \theta$. Then $\sup^+(\mathrm{dom}(f_j)) \leq \theta < \sup^+(\mathrm{ran}(f_j))$ since $\mathrm{dom}(f_0) = \mathrm{dom}(f)$ and $\mathrm{ran}(f_i) = \mathrm{dom}(f_{i+1})$ for every $i < j$. However, from (c) it follows that $\sup^+(\mathrm{ran}(f_j)) \leq \sup^+(\mathrm{dom}(f_k)) < \sup^+(\mathrm{ran}(f_k))$ for all $k \in \{j+1, \dots, \mathrm{lh}(\bar{f}) - 1\}$, and so $\sup^+(\mathrm{ran}(f)) = \sup^+(\mathrm{ran}(f_{\mathrm{lh}(\bar{f})-1})) > \theta$, a contradiction.

e) By applying (b) to ${f_n}^{-1}$, for $n = \mathrm{lh}(\bar{f}) - 1$, we get that $\sup^+(\mathrm{dom}(f_n)) \geq \sup^+(\mathrm{ran}(f_n)) = \mu$. By (d), $\sup^+(\mathrm{dom}(f_n)) \leq \mu$. Hence $\sup^+(\mathrm{dom}(f_n)) = \mu$. In the same way it can be shown $\sup^+(\mathrm{dom}(f_i)) = \mu$ for all $i \leq n$.

Suppose $\mathrm{dom}(f) = \mu$. By Lemma 3.14, $\mathrm{ind}(f_i)$ is a limit point, say $u_i \in U$, and $\sup^+(\mathrm{ran}(f_i)) = \mu = \mathrm{dom}(p_{u_i}) = \mathrm{ran}(p_{u_i})$ for every $i \leq n$. However, when $n > 1$, $f_0 \subseteq p_{u_0}$ together with $\mathrm{dom}(f_0) = \mathrm{dom}(f) = \mu = \mathrm{dom}(p_u)$ imply that $f_0 = p_{u_0}$ and $\mathrm{ran}(f_0) = \mu = \mathrm{dom}(f_1)$. A similar reasoning shows $\mathrm{dom}(f_i) = \mathrm{ran}(f_i) = \mu$ for every $i \leq n$. $\blacksquare_{3.15}$

Lemma 3.16

a) *Suppose* $\bar{f}, \bar{g} \in \bar{\mathcal{F}}$ *are such that* $\mathrm{dom}(f) = \mathrm{dom}(g) = \{\xi\}$, $f(\xi) = g(\xi)$, *and* f_i *is increasing for every* $i < \mathrm{lh}(\bar{f})$. *Then* $\mathrm{lh}(\bar{f}) \leq \mathrm{lh}(\bar{g})$ *and for* $k = \mathrm{lh}(\bar{g}) - \mathrm{lh}(\bar{f})$, *both* $f_i = g_{k+i}$ *and* $\mathrm{beg}(f_i) = \mathrm{beg}(g_{k+i})$ *hold for every* $i < \mathrm{lh}(\bar{f})$. *Moreover, if* $\mathrm{beg}(g) = \mathrm{beg}(f)$ *or for every* $j < \mathrm{lh}(\bar{g})$, g_j *is increasing, then* $\mathrm{lh}(\bar{f}) = \mathrm{lh}(\bar{g})$.

b) *Suppose* $q \in \mathcal{F}_1$, $\xi < \zeta \in \mathrm{dom}(q)$, *and that both* $q{\restriction}\{\xi\}$ *and* $q{\restriction}\{\zeta\}$ *are increasing. If there is no* $d \in \mathcal{E}$ *with* $\{\xi, \zeta\} \subseteq \mathrm{dom}(d)$ *then* $\mathrm{card}(\xi) < \beth_{\mathrm{ref}(q{\restriction}\{\xi\})} \leq \mathrm{card}(\zeta)$.

c) *Suppose* $\langle q_1, q_2 \rangle \in \bar{\mathcal{F}}$ *and* $\sup^+(\mathrm{dom}(q_1)) < \sup^+(\mathrm{ran}(q_1))$. *Then* $\mathrm{ind}(q_1)$, $\mathrm{ind}(q_2)$ *are successors, the functions* $d^1 = c^{\mathrm{ind}(q_1)}{\restriction}\mathrm{dom}(q_1)$, $d^2 = c^{\mathrm{ind}(q_2)}{\restriction}\mathrm{dom}(q_2)$ *satisfy the demand that they both are increasing, and* $\mathrm{dom}(q_2) \smallsetminus \min\mathrm{ran}(d^1) \subseteq \mathrm{dom}(d^2)$.

d) *Suppose* $\bar{f} \in \bar{\mathcal{F}}$ *and* $\sup^+(\mathrm{dom}(f_0)) < \sup^+(\mathrm{ran}(f_0))$. *Then for every* $i < \mathrm{lh}(\bar{f})$, $\mathrm{ind}(f_i)$ *is successor,* $c^{\mathrm{ind}(f_i)}$ *is increasing, and particularly, for* $d = c^{\mathrm{ind}(f_0)}{\restriction}\mathrm{dom}(f_0)$ *and for every* $i \in \{1, \ldots, \mathrm{lh}(\bar{f}) - 1\}$, $\mathrm{dom}(f_i) \smallsetminus f_{<i}\big(\min\mathrm{ran}(d)\big) \subseteq c^{\mathrm{ind}(f_i)}$.

e) *Let* $\bar{f}$ *be as in* (d). *For* $\bar{u} = \langle \mathrm{ind}(f_i) \mid i < \mathrm{lh}(\bar{f}) \rangle$ *and for every* θ *with* $\sup^+(\mathrm{dom}(f)) < \theta \leq \beth_{\mathrm{ref}(d)}$, *the pair* $\langle \bar{u}, \theta \rangle$ *is in* Seq *and the sequence* $\bar{g}^{\bar{u},\theta}$ *is in* $\bar{\mathcal{F}}$ *(see Definition 3.11). Furthermore, if* $\mathrm{end}(f)$ *is a function with a constant value, then* $\mathrm{end}(g^{\bar{u},\theta}) \supseteq \mathrm{end}(f)$ *is also a constant function.*

f) *For every* $\bar{f}$ *in* $\bar{\mathcal{F}}$ *there is* $\xi \in \mathrm{dom}(f)$ *with* $\mathrm{ran}(f) \subseteq f(\xi) + \sup^+(\mathrm{dom}(f))$.

g) *Suppose* $\bar{g} \in \bar{\mathcal{F}}$, $\mathrm{dom}(g)$ *is a cardinal* μ, $\mathrm{ind}(g_0) = u_0$ *is successor, and* c^{u_0} *is increasing. Assume* $\bar{h} \in \bar{\mathcal{F}}$ *is such that* $\mathrm{dom}(h) = \{\xi, \xi'\} \subseteq \mu$,

$\xi \in \mathrm{dom}(c^{u_0})$, $h(\xi) = g(\xi)$, *and* $\mathrm{beg}(h) \subseteq \mathrm{beg}(g)$. *Then either* $h(\xi') \in \mathrm{ran}(g)$, *or otherwise,* $h(\xi') \geq h(\xi) + \mu$.

Proof. a) Denote $\mathrm{lh}(\bar{f}) - 1$ by n and $\mathrm{lh}(\bar{g}) - 1$ by m. If g_m was decreasing, then, by applying Lemma 3.15(c) to the sequence $\langle {g_m}^{-1}, \ldots, {g_0}^{-1} \rangle$ in $\bar{\mathcal{F}}$, g_i should be decreasing for every $i \leq m$ and $\sup^+(\mathrm{dom}(g)) > \sup^+(\mathrm{ran}(g_m)) = \sup^+(\mathrm{ran}(f_n)) > \sup^+(\mathrm{dom}(f)) = \xi{+}1$ contrary to the assumption $\mathrm{dom}(f) = \mathrm{dom}(g) = \{\xi\}$. Thus g_m is increasing. Since $\mathrm{ran}(f_n) = \mathrm{ran}(g_m)$, $f_n = g_m$ by Fact 3.8(a). By Lemma 3.15(a), $\mathrm{beg}(f_n) = \mathrm{beg}(g_m)$. When $n > 0$, $\mathrm{ran}(f_{n-1}) = \mathrm{dom}(f_n) = \mathrm{dom}(g_m) = \mathrm{ran}(g_{m-1})$. Hence we can repeat the same argument and we get that $f_{n-i} = g_{m-i}$ and $\mathrm{beg}(f_{n-i}) = \mathrm{beg}(g_{m-i})$ for every $i \leq \min\{m, n\}$. However $m \geq n$ since otherwise $\mathrm{dom}(g) = \{f_{<n-m}(\xi)\} \neq \{\xi\}$.

If $m > n$ and $\mathrm{beg}(g) = \mathrm{beg}(f)$ then $g_{\leq m-n-1}(\xi) = \xi$ and $\mathrm{end}(g_{m-n-1}) = \mathrm{beg}(g_{m-n}) = \mathrm{beg}(f_0) = \mathrm{beg}(f) = \mathrm{beg}(g) = \mathrm{beg}(g_0)$ contrary to the minimality of $\bar{g}$.

If $m > n$ and g_j is increasing, for every $j < \mathrm{lh}(\bar{g})$, then $\mathrm{ran}(g_{m-n-1}) = \mathrm{dom}(g_{m-n}) = \mathrm{dom}(f_0) = \{\xi\}$ and $\mathrm{dom}(g) = \{{g_0}^{-1} \circ \ldots \circ {g_{m-n-1}}^{-1}(\xi)\} \neq \{\xi\}$, a contradiction.

b) Let $u \trianglelefteq \mathrm{ind}(q)$ be the smallest element with $\xi \in \mathrm{dom}(p_u)$, and $v \trianglelefteq \mathrm{ind}(q)$ be the smallest element with $\zeta \in \mathrm{dom}(p_v)$. Then u and v are successors, $u \trianglelefteq v$, $\xi \in \mathrm{dom}(c^u)$, $\zeta \in \mathrm{dom}(c^v)$, and $q \restriction \{\xi, \zeta\} \subseteq c^u \cup c^v \subseteq p_u \cup c^v$. Assume $c^u \neq c^v$. Then $u \triangleleft v$. Since c^u is increasing $\mathrm{dom}(p_u) = \beth_\beta$ where $\beta = \mathrm{ref}(c^u)$. Because $\zeta \in \mathrm{dom}(c^v) \setminus \mathrm{dom}(c^u)$ we have $\mathrm{dom}(c^v) \cap \beth_\beta = \emptyset$. So $\mathrm{card}(\xi) < \beth_\beta \leq \mathrm{card}(\zeta)$.

c) follows that $\sup^+(\mathrm{dom}(q_2)) < \sup^+(\mathrm{ran}(q_2))$. The elements $\mathrm{ind}(q_1)$ and $\mathrm{ind}(q_2)$ are successors by Lemma 3.14. If $q_2(\xi) = \xi$ then q_2 should be identity contrary to Fact 3.13(b). So both d^1 and d^2 are increasing. If for some $\xi \in \mathrm{ran}(d^1)$, $q_2 \restriction \{\xi\}$ is decreasing, then $(d^1)^{-1} \restriction \{\xi\} = q_2 \restriction \{\xi\}$, and as in the proof of Lemma 3.15(c), $(q_1)^{-1} = q_2$ and $\mathrm{beg}(q_1) = \mathrm{end}(q_2)$ contrary to the minimality of the sequence $\langle q_1, q_2 \rangle$. Thus $q_2 \restriction \{\xi\}$ is increasing for all $\xi \in \mathrm{ran}(d^1)$. Now $\mathrm{card}(\xi) = \mathrm{card}(\min \mathrm{ran}(d^1))$ for each $\xi \in \mathrm{ran}(d^1)$. By (b), there is $e \in \mathcal{E}$, with $e \subseteq q_2$ and $\mathrm{dom}(e) = \mathrm{ran}(d^1)$. Because $\mathrm{ran}(d^1) = \mathrm{dom}(e)$ is an end segment of $\mathrm{ran}(q_1) = \mathrm{dom}(q_2)$, $\mathrm{dom}(q_2) \setminus \min \mathrm{ran}(d^1) = \mathrm{dom}(e)$. Since e is increasing, it follows from the definition of d^2 that $e \subseteq d^2$.

d) The claim follows from (c) by induction on $i < \mathrm{lh}(\bar{f})$.

e) It suffices to show that $\langle \bar{u}, \theta \rangle \in \mathrm{Seq}$ since then the minimality of $\bar{g}^{\bar{u},\theta}$ follows from the fact that $\bar{f}$ is in $\bar{\mathcal{F}}$. Let β_i denote $\mathrm{ref}(c^{u_i})$ for every $i <$

$\mathrm{lh}(\bar{u}) = \mathrm{lh}(\bar{f})$. Abbreviate $\min \mathrm{dom}(d)$ by ξ. By (d), $\mathrm{ran}(p_{u_i}) \subseteq \beth_{(\beta_i)+1} \leq \beth_{\beta_{(i+1)}} = \mathrm{dom}(p_{u_{i+1}})$ and $f_{\leq i}(\xi) \in \beth_{\beta_i+1} \smallsetminus \beth_{\beta_i}$ for every $i < \mathrm{lh}(\bar{u}) - 1$. So $\langle \bar{u}, \theta \rangle$ satisfies Definition 3.11(c). From $\langle \bar{u}, \mathrm{dom}(f) \rangle \in \mathrm{Seq}$ it follows that $\mathrm{fun}_2(u_i) \restriction \big(f_{\leq i}(\xi) + 1\big) = \mathrm{fun}_1(u_{i+1}) \restriction \big(f_{\leq i}(\xi) + 1\big)$ for all $i < \mathrm{lh}(\bar{u}) - 1$. These equations together with Definition 3.5 ensure that both of the functions $\mathrm{fun}_2(u_i)$ and $\mathrm{fun}_1(u_{i+1})$, for $i < \mathrm{lh}(\bar{u}) - 1$, have the same constant value on the interval $\beth_{\beta_i+1} \smallsetminus \beth_{\beta_i}$. Hence the pair $\langle \bar{u}, \theta \rangle$ satisfies Definition 3.11(d), too. Similarly, the latter claim, concerning $\mathrm{end}(f)$, is a consequence of the facts that for $n = \mathrm{lh}(\bar{u}) - 1$, $f(\xi) \in \mathrm{ran}(g^{\bar{u},\theta}) = \mathrm{ran}(g_n^{\bar{u},\theta}) \subseteq \beth_{\beta_n+1} \smallsetminus \beth_{\beta_n}$ and $\mathrm{fun}_2(u_n)$ is a constant function on the interval $\beth_{\beta_n+1} \smallsetminus \beth_{\beta_n}$.

f) Abbreviate $\sup^+(\mathrm{dom}(f))$ by θ, $\mathrm{lh}(\bar{f}) - 1$ by n, and for every $i \leq n$, $\mathrm{ind}(f_i)$ by u_i. If $\sup^+(\mathrm{ran}(f)) \leq \theta$ there is nothing to prove. So assume $\sup^+(\mathrm{ran}(f)) > \theta$. There must be the smallest index $j \leq n$ satisfying $\sup^+(\mathrm{dom}(f_j)) \leq \theta < \sup^+(\mathrm{ran}(f_j))$, u_j is a successor, and $c^{u_j} \restriction \mathrm{dom}(f_j)$, abbreviated by d, is increasing. Let ξ be $\min \mathrm{dom}(d)$. Then for all $\zeta \in \mathrm{dom}(f_j) \smallsetminus \mathrm{dom}(d)$, $f_j(\zeta) < f_j(\xi)$, by the definition of c^{u_j}. Besides, $\mathrm{dom}(f_j) \subseteq \theta$ together with Fact 3.8(c) ensure that for all $\zeta \in \mathrm{dom}(d)$, $f_j(\zeta) - f_j(\xi) = d(\zeta) - d(\xi) = \zeta - \xi < \theta$. So the claim holds in case $j = n$.

Suppose $n > j$. From (d) it follows that for every $i \in \{j+1, \ldots, n\}$, u_i is a successor, c^{u_i} is increasing, and $\mathrm{dom}(f_i) \smallsetminus f_{<i}(\xi) \subseteq \mathrm{dom}(c^{u_i})$. For all $\zeta \in \mathrm{dom}(f_j) \smallsetminus \mathrm{dom}(d)$, $f(\zeta) < f(\xi)$ since $f_j(\zeta) < f_j(\xi)$ and the property "$f_i \restriction \{\xi\}$ is increasing for every $i \in \{j+1, \ldots, n\}$" implies $f_{\leq i}(\zeta) < f_{\leq i}(\xi)$ for every $i \in \{j+1, \ldots, n\}$. Suppose $\zeta \in \mathrm{dom}(d)$, $i \in \{j+1, \ldots, n\}$, and $f_{<i}(\xi) < f_{<i}(\zeta) < f_{<i}(\xi) + \theta$. Then $\{f_{<i}(\xi), f_{<i}(\zeta)\} \subseteq \mathrm{dom}(c^{u_i})$ and by Fact 3.8(c),

$$f_{\leq i}(\zeta) - f_{\leq i}(\xi) = c^{u_i}(f_{<i}(\zeta)) - c^{u_i}(f_{<i}(\xi)) = f_{<i}(\zeta) - f_{<i}(\xi) < \theta.$$

The claim follows from the fact that $\mathrm{ran}(f) \smallsetminus f(\xi) = \mathrm{ran}(f_n) \smallsetminus f_{\leq n}(\xi) \subseteq \mathrm{ran}(c^{u_n})$ (remember $\mathrm{dom}(f_n) \smallsetminus f_{<n}(\xi) \subseteq \mathrm{dom}(c^{u_n})$ and c^{u_n} is increasing).

g) Denote $\mathrm{lh}(\bar{g}) - 1$ by n, $\mathrm{lh}(\bar{h}) - 1$ by m and for each $i \leq m$ abbreviate $h_{<i}(\xi)$ by ξ_i and $h_{<i}(\xi')$ by ξ'_i. Write ξ_{m+1} for $h(\xi)$ and ξ'_{m+1} for $h(\xi')$. Note that by (d), for every $i \leq n$ and for $u_i = \mathrm{ind}(g_i)$, u_i is a successor, c^{u_i} is increasing, and $g_i \restriction \{g_{<i}(\xi)\} = c^{u_i} \restriction \{g_{<i}(\xi)\}$.

There exists the smallest index $j \leq m$ with $h_j \restriction \{\xi_j\} = g_0 \restriction \{\xi\}$, because otherwise for the minimal reduct $\bar{d}$ of the sequence $\langle h_i \restriction \{\xi_i\} \mid i \leq m \rangle$ (see Fact 3.13), $\bar{d}$ differs from the minimal sequence $\bar{e} = \langle g_i \restriction \{g_{<i}(\xi)\} \mid i \leq n \rangle$ and $e_n \circ \ldots \circ e_0 = d_{\mathrm{lh}(\bar{d})-1} \circ \ldots \circ d_0$ contrary to (a). We have two cases to consider:

1) $\xi'_j \geq \mu$;

2) $\xi'_j < \mu$.

1) Suppose first that $\xi'_j \geq \mu$. Note that $\mu > \xi = \xi_j$. Note that $h_j(\xi'_j) \neq \xi'_j$ since otherwise also $h_j(\xi_j) = \xi_j$. The function $h_j \restriction \{\xi'_j\}$ must be increasing, namely otherwise, we reach a contradiction in the following manner. Assume $h_j \restriction \{\xi'_j\}$ is decreasing. There are two subcases:

i) Assume that $\sup^+(\mathrm{dom}(h_j)) > \sup^+(\mathrm{ran}(h_j))$ or $\sup^+(\mathrm{dom}(h_{j-1})) > \sup^+(\mathrm{ran}(h_{j-1}))$.

ii) By Lemma 3.14, both $\sup^+(\mathrm{dom}(h_j)) \neq \sup^+(\mathrm{ran}(h_j))$ and $\sup^+(\mathrm{dom}(h_{j-1})) \neq \sup^+(\mathrm{ran}(h_{j-1}))$ hold. So suppose $\sup^+(\mathrm{dom}(h_j)) < \sup^+(\mathrm{ran}(h_j))$ and $\sup^+(\mathrm{dom}(h_{j-1})) < \sup^+(\mathrm{ran}(h_{j-1}))$.

i) It would follow from the assumption $\xi'_j \geq \mu$ and by applying (d) to the sequence $\langle {h_j}^{-1}, \dots, {h_0}^{-1} \rangle$ or $\langle {h_{j-1}}^{-1}, \dots, {h_0}^{-1} \rangle$, that $\sup^+(\mathrm{dom}(h)) = \sup^+(\mathrm{dom}(h_0)) > \mu$, a contradiction.

ii) The function $h_{j-1} \restriction \{\xi'_{j-1}\}$ is increasing, since otherwise,

$$\sup^+(\mathrm{dom}(h_{j-1})) \geq \xi'_{j-1} + 1 \geq h_{j-1}(\xi'_{j-1}) + 1 = \xi'_j + 1 = \sup^+(\mathrm{ran}(h_{j-1})).$$

Let β be $\mathrm{ref}(h_{j-1} \restriction \{\xi'_{j-1}\})$. If $\xi_{j-1} \geq \beth_\beta$, then $\xi_{j-1} > \xi'_{j-1}$ and $h(\xi_{j-1}) \neq \xi_{j-1}$. Moreover, $\mathrm{ref}(h_{j-1} \restriction \{\xi_{j-1}\}) > \beta$ and $\sup^+(\mathrm{dom}(h_{j-1})) = \xi_{j-1} + 1 > \beth_{\beta+1} > \xi'_j + 1 = \sup^+(\mathrm{ran}(h_{j-1}))$, a contradiction. On the other hand, if $\xi_{j-1} < \beth_\beta$, then it follows from the assumption $\xi'_j > h_j(\xi'_j)$ that $h_{j-1} \restriction \{\xi'_{j-1}\} = \left(h_j \restriction \{\xi'_j\}\right)^{-1}$. By Lemma 3.15(a), $h_j = {h_{j-1}}^{-1}$ and $\mathrm{beg}(h_{j-1}) = \mathrm{end}(h_j)$ contrary to the minimality of $\bar{h}$.

Hence $h_j \restriction \{\xi'_j\}$ is increasing, and by (d), $\mathrm{ind}(h_i)$, abbreviated by v_i, is successor, c^{v_i} is increasing, and $\xi'_i \in \mathrm{dom}(c^{v_i})$ for every $i \in \{j, \dots, m\}$. We show by induction on $i \in \{j, \dots, m\}$ that $\xi_i + \mu \leq \xi'_i$ where $+$ is the ordinal addition.

Since $\xi_j = \xi < \mu$, $\xi'_j \geq \mu$, and μ is cardinal, we have $\xi_j + \mu \leq \xi'_j$. Suppose $i < m$ and $\xi_i + \mu \leq \xi'_i$. If $h_i(\xi_i) = \xi_i = \xi_{i+1}$ then $\xi_{i+1} + \mu = \xi_i + \mu \leq \xi'_i < h_i(\xi'_i) = \xi'_{i+1}$.

If $\xi_i \in \mathrm{dom}(c^{v_i})$, then $\xi'_{i+1} = c^{v_i}(\xi'_i)$, $\xi_{i+1} = c^{v_i}(\xi_i)$, and $\xi'_{i+1} - \xi_{i+1} = c^{v_i}(\xi'_i) - c^{v_i}(\xi_i) = \xi'_i - \xi_i \geq \mu$.

If $\xi_i \notin \mathrm{dom}(c^{v_i})$, then the reflection point of $h_i \restriction \{\xi_i\}$, say β, is smaller than $\mathrm{ref}(c^{v_i})$ by the definition of c^{v_i}. Since $\xi_{i+1} \leq \beth_{\beta+1} \leq \beth_{\mathrm{ref}(c^{v_i})}$ and $\mu < \beth_{\mathrm{ref}(c^{v_i})}$, it follows that $\xi_{i+1} + \mu < \beth_{\mathrm{ref}(c^{v_i})} < \xi'_{i+1}$.

2) Suppose then that $\xi'_j < \mu$. Abbreviate $\mathrm{ref}(c^{u_i})$, for $i \leq n$, by γ_i. Since $h_j \restriction \{\xi_j\} = c^{u_0} \restriction \{\xi\} = g_0 \restriction \{\xi\}$ is increasing and $\{\xi_j, \xi'_j\} \subseteq \mu = \mathrm{dom}(g_0) \leq \beth_{\gamma_0}$, we get by Lemma 3.15(a), that $\mathrm{beg}(h_j) \subseteq \mathrm{beg}(g_0)$ and $h_j \restriction \{\xi'_j\} = g_0 \restriction \{\xi'_j\}$. By (d), $h_i \restriction \{\xi_i\}$ are increasing for all $i \in \{j, \ldots, m\}$. It follows from $\xi = \xi_j$ and $h(\xi_j) = h(\xi) = g(\xi)$ together with (a), that $\langle h_i \restriction \{\xi_i\} \mid j \leq i \leq m \rangle = \langle g_k \restriction \{g_{<k}(\xi)\} \mid k \leq n \rangle$.

To show that $h(\xi') = h_m \circ \ldots \circ h_j(\xi'_j) \in \mathrm{ran}(g)$ we prove by induction on $k \leq n$ that $h_{j+k} \restriction \{\xi'_{j+k}\} = g_k \restriction \{\xi'_{j+k}\}$. Note that $m = j + n$ and it is possible that $\xi'_j \neq \xi'$. We already proved the case $k = 0$. Suppose $k > 0$ and for every $i < k$ the subclaim holds. Then $\{\xi_{j+k}, \xi'_{j+k}\} = \mathrm{dom}(h_{j+k}) \subseteq \mathrm{ran}(g_{k-1}) = \mathrm{dom}(g_k)$. Since $h_{j+k} \restriction \{\xi_{j+k}\} = g_k \restriction \{g_{<k}(\xi)\}$ is increasing and $\xi'_{j+k} \in \mathrm{dom}(g_k) \subseteq \beth_{\gamma_k}$, we get by Lemma 3.15(a), that $h_{j+k} \restriction \{\xi'_{j+k}\} = g_k \restriction \{\xi'_{j+k}\}$. $\blacksquare_{3.16}$

Lemma 3.17

a) *Suppose $p, q \in \mathcal{F}_1$ are such that* $\mathrm{ind}(p)$ *is a limit,* $\mathrm{dom}(p) = \mathrm{dom}(q) = X$*, and the set* $Y = \{\zeta \in X \mid p \restriction \{\zeta\} = q \restriction \{\zeta\}\}$ *is unbounded in* X*. Then* $\mathrm{ind}(q) = \mathrm{ind}(p)$*, and particularly,* $p = q$*,* $\mathrm{beg}(p) = \mathrm{beg}(q)$*, and* $\mathrm{end}(p) = \mathrm{end}(q)$*.*

b) *Suppose $\bar{f}$ in $\bar{\mathcal{F}}$ and the set* $I_0 = \{\xi \in \mathrm{dom}(f_0) \mid \xi < f_0(\xi)\}$ *is unbounded in* $\mathrm{dom}(f_0)$*. Then there is an end segment J of I_0 such that for every $\xi \in J$ and $i < \mathrm{lh}(\bar{f})$, $f_i \restriction \{f_{<i}(\xi)\}$ is increasing.*

c) *Suppose $\bar{f}, \bar{g} \in \bar{\mathcal{F}}$ and $n < \omega$ are such that* $\mathrm{lh}(\bar{f}) = \mathrm{lh}(\bar{g}) = n$*,* $\mathrm{dom}(f) = \mathrm{ran}(f)$ *is a cardinal, and $f \subseteq g$. Then* $\mathrm{ind}(f_i) \trianglelefteq \mathrm{ind}(g_i)$ *and $f_i \subseteq g_i$ holds for every $i < n$.*

d) *Suppose $\bar{\alpha}$ is an increasing sequence $\langle \alpha_l \mid l < \omega \rangle$ of ordinals below κ such that* $\sup^+(\bar{\alpha})$ *is a cardinal. Then for every $\bar{f} \in \bar{\mathcal{F}}$ such that* $\mathrm{dom}(f) = \{\alpha_l \mid l < \omega\}$*, there are infinitely many indices $l < \omega$ with $f(\alpha_l) \neq \alpha_l$.*

Proof. a) Let u be $\mathrm{ind}(p)$ and v be $\mathrm{ind}(q)$. We may assume that $p(\zeta) = q(\zeta) \neq \zeta$ for all $\zeta \in Y$. Let Z be the set $\{\min\{\zeta, p(\zeta)\} \mid \zeta \in Y\}$. By

Lemma 3.15(a), $u\restriction\xi = v\restriction\xi$ for every $\xi \in Z$. Since Y is unbounded in X and $\mathrm{ind}(p)$ is a limit, also Z is unbounded in X. By Lemma 3.14, $\sup^+(X) = \sup^+(\mathrm{dom}(p)) = \mathrm{dom}(p_u) = \beth_{\mathrm{ord}(u)}$. Hence, as in the proof of Fact 3.6, $u \trianglelefteq v$. By Definition 3.12(a), $u = v$. Because p and q have the common domain X, it follows that $p = q$, $\mathrm{beg}(p) = \mathrm{beg}(q)$ and $\mathrm{end}(p) = \mathrm{end}(q)$.

b) If $\mathrm{ind}(f_0)$ is a successor then $\sup^+(\mathrm{dom}(f_0)) < \sup^+(\mathrm{ran}(f_0))$, and the claim follows from Lemma 3.16(d). Suppose $\mathrm{ind}(f_0)$ is a limit. By Lemma 3.14, $\sup^+(\mathrm{dom}(f_0))$ is a cardinal. Suppose, contrary to the claim, that $j < n$ is the smallest index for which there is unbounded $J \subseteq I_0$ such that for every ζ in the set $Y = \{f_{<j}(\xi) \mid \xi \in J\}$, $f_j\restriction\{\zeta\}$ is increasing and $f_{j+1}\restriction\{f_j(\zeta)\}$ is not increasing. Then $\mathrm{ind}(f_i)$ is a limit for every $i \leq j$, since otherwise the existence of the chosen j contradicts Lemma 3.16(d). Therefore $\sup^+(\mathrm{dom}(f_j))$ is the cardinal $\sup^+(\mathrm{dom}(f_0))$, and necessarily Y is unbounded in $\mathrm{dom}(f_j)$. So we may assume that $f_{j+1}\restriction\{f_j(\zeta)\}$ is decreasing for every $\zeta \in Y$. Since $f_j\restriction\{\zeta\}$ must equal $\left(f_{j+1}\restriction\{f_j(\zeta)\}\right)^{-1}$ for every $\zeta \in Y$ it follows from (a) that $f_j = {f_{j+1}}^{-1}$ and $\mathrm{beg}(f_j) = \mathrm{end}(f_{j+1})$ contrary to the minimality of $\bar{f}$.

c) In the case $n = 1$ the claim is proved in (a). Assume $n > 1$. Let θ be the cardinal $\mathrm{dom}(f) = \mathrm{ran}(f)$. For each $i < n$, $\mathrm{dom}(f_i) = \mathrm{ran}(f_i) = \theta$ by Lemma 3.15(e). By Lemma 3.14, $\mathrm{ind}(f_i)$ is a limit point and $f_i = p_{\mathrm{ind}(f_i)}$ for every $i < n$. Denote the set $\{\xi < \theta \mid f_0(\xi) > \xi\}$ by I_0. Then I_0 must be unbounded in θ by Definition 3.9. For each $i < n-1$ define I_{i+1} to be $\{\xi \in f_i[I_i] \mid f_{i+1}(\xi) > \xi\}$.

By (b), there is an end segment K of I_0 satisfying that for every $i < n-1$, $f_{\leq i}[K]$ is an end segment of I_{i+1}. Now $\mathrm{lh}(\bar{f}) = \mathrm{lh}(\bar{g}) = n$ and $f(\xi) = g(\xi)$ together with Lemma 3.16(a) imply that for all $\xi \in K$ and $i < n$, the equations $f_{\leq i}(\xi) = g_{\leq i}(\xi)$ hold. Since K is unbounded in θ and for each $i < n$, $f_i\restriction f_{<i}[K]$ is increasing, also $f_{\leq i}[K]$ is unbounded in θ for every $i < n$. By (a), $\mathrm{ind}(f_i) \trianglelefteq \mathrm{ind}(g_i)$ and $f_i = f_i\restriction\theta = g_i\restriction\theta$ for every $i < n$.

d) Let θ be the cardinal $\sup^+(\bar{\alpha})$ and let n denote the length of $\bar{f}$. For every $l < \omega$ and $i < n$ write d_i^l for the function $f_i\restriction\left\{f_{<i}(\alpha_l)\right\}$. For every $i < n$ define I_i to be the set $\{l < \omega \mid d_i^l \text{ is increasing}\}$.

Suppose, contrary to the claim, that there is $m < \omega$ such that $f(\alpha_l) = \alpha_l$ hold for all $l \in \omega \setminus m$. By (b), I_0 is finite. There must be the smallest $j \in \{1, \ldots, n-1\}$ such that I_j is infinite. By Lemma 3.15(b), $\sup^+(\mathrm{dom}(f_0)) = \sup^+(\mathrm{ran}(f_0))$ and so $\mathrm{ind}(f_0)$ is a limit. Since $\mathrm{dom}(f_{i+1}) = \mathrm{ran}(f_i)$ for all $i < j$, we get by applying Lemma 3.15(b) repeatedly, that $\mathrm{ind}(f_i)$ are limit points for all $i < j$. By the choice of j, there is an end segment J of ω such that $\min J \geq m$ and d_i^l is decreasing for all $l \in J$ and $i < j$ (d_i^l cannot be

identity for unbounded many $l < \omega$). The set $Y = \{f_{<j}(\alpha_l) \mid l \in J\}$ is unbounded in $\mathrm{dom}(f_j)$.

If $\mathrm{ind}(f_j)$ is a successor, then $Y \cap \mathrm{dom}(c^{\mathrm{ind}(f_j)})$ is infinite, and by Lemma 3.16(d), $f(\alpha_l) > \theta > \alpha_l$ for infinitely many $l \in J$, a contradiction. Hence $\mathrm{ind}(f_j)$ is a limit. By (b), there is an end segment K of J such that for every $l \in J$ and $k \in \{j, \ldots, n-1\}$, d^l_k is increasing. For every $l \in K$, $f(\alpha_l) = \alpha_l$ holds, and thus the compositions $(d^l_0)^{-1} \circ \ldots \circ (d^l_{j-1})^{-1}$ and $d^l_{n-1} \circ \ldots \circ d^l_j$ are equal. Since $\mathrm{end}(f_{j-1}) = \mathrm{beg}(f_j)$ and the sequences $\langle (d^l_{j-1})^{-1}, \ldots, (d^l_0)^{-1} \rangle$ and $\langle d^l_j, \ldots, d^l_{n-1} \rangle$ are in $\bar{\mathcal{F}}$ (in both of the sequences all the functions are increasing), it follows from Lemma 3.16(a), that these sequences are equal. Particularly, $d^l_j = (d^l_{j-1})^{-1}$ for every $l \in K$. From (a) it would follow that $f_{j-1} = f_j$ and $\mathrm{beg}(f_{j-1}) = \mathrm{end}(f_j)$, contrary to the minimality of $\bar{f}$. $\blacksquare_{3.17}$

Definition 3.18 *We define D to be the following closed and unbounded subset of κ:*

$$\Big\{\mu \in \kappa \smallsetminus (\lambda + 1) \mid \big\langle V_\mu, \in, \pi \cap V_\mu, X \cap V_\mu, Y \cap V_\mu \big\rangle \prec \big\langle V_\kappa, \in, \pi, X, Y \big\rangle \Big\},$$

where π is the function from Definition 3.4, $X = \{\langle p_u, \mathrm{beg}(p_u), \mathrm{end}(p_u)\rangle \mid u \in U\}$, and $Y = \{\langle v, p_v, \mathrm{beg}(p_v), \mathrm{end}(p_v)\rangle \mid v \in U\}$.

Note that for all $\mu \in D$, $\beth_\mu = \mu$.

Definition 3.19 *For all $\mu \in D \cup \{\kappa\}$ and $\eta, \nu \in {}^{\mu}2$ define:*

$$\begin{aligned} \bar{\mathcal{F}}[\eta,\nu] &= \Big\{\bar{f} \in \bar{\mathcal{F}} \mid \mathrm{ind}(f_i) \in U[<\mu] \textit{ for all } i < \mathrm{lh}(\bar{f}), \\ &\qquad \mathrm{beg}(f) \subseteq \eta, \textit{ and } \mathrm{end}(f) \subseteq \nu\Big\}; \\ \mathcal{F}_1[\eta,\nu] &= \Big\{f \mid \bar{f} \in \bar{\mathcal{F}}[\eta,\nu] \textit{ and } \mathrm{lh}(\bar{f}) = 1\Big\}. \end{aligned}$$

Lemma 3.20 *Suppose $\mu \in D \cup \{\kappa\}$.*

a) *For every $u \in U$, if u is a limit point or a successor in U^2_λ, then $p_u \in V_\mu$ implies $u \in U[<\mu]$. For all successors $u \in U^1$, if p_u is in V_μ then there is $v \in U[<\mu]$ such that $p_v = p_u$, $\mathrm{beg}(p_v) = \mathrm{beg}(p_u)$, and $\mathrm{end}(p_v) = \mathrm{end}(p_u)$.*

b) *For every* $v \in U[<\mu]$ *and* $\gamma < \mu$, *there is* $\alpha \in (\mathrm{Suc}^+ \cap \mu) \smallsetminus (\gamma+1)$ *such that for every* $\eta', \nu' \in {}^{\beth_\alpha}2$ *with* $\mathrm{fun}_1(v) \subseteq \eta'$ *and* $\mathrm{fun}_2(v) \subseteq \nu'$, *we can find* $u^{\eta',\nu'} \in U^2_\lambda[\alpha]$ *satisfying that* $\mathrm{fun}_1(u^{\eta',\nu'}) = \eta'$, $\mathrm{fun}_2(u^{\eta',\nu'}) = \nu'$, *and* $u^{\eta',\nu'}$ *is a successor of* v.

c) *If* $\eta, \nu \in {}^{\mu}2$ *and* $q \in \mathcal{F}_1[\eta,\nu]$ *is such that for* $v = \mathrm{ind}(q)$ *both* $\mathrm{fun}_1(v) \subseteq \eta$ *and* $\mathrm{fun}_2(v) \subseteq \nu$ *hold, then there is* $u \in U[<\mu]$ *satisfying that* $\mathrm{ind}(q) \trianglelefteq u$ *(implying* $q \subseteq p_u$*),* $p_u \in \mathcal{F}_1[\eta,\nu]$, *and* $\theta \subseteq \mathrm{dom}(p_u) \cap \mathrm{ran}(p_u)$.

d) *Suppose* $\eta, \nu \in {}^{\mu}2$, $q \in \mathcal{F}_1[\eta,\nu]$, *and* $\theta < \mu$. *There is* $\bar{f} \in \bar{\mathcal{F}}[\eta,\nu]$ *such that* $q \subseteq f$ *and* $\theta \subseteq \mathrm{dom}(f) \cap \mathrm{ran}(f)$.

e) *Suppose* $\eta, \nu \in {}^{\mu}2$, $\bar{f} \in \bar{\mathcal{F}}[\eta,\nu]$, *and* $\theta < \mu$. *There is* $\bar{g} \in \bar{\mathcal{F}}[\eta,\nu]$ *with* $g \supseteq f$ *and* $\theta \subseteq \mathrm{dom}(g) \cap \mathrm{ran}(g)$.

Proof. The properties (a)–(c) are straightforward consequences of the definition of the functions p_u. We sketch proofs of the rest of the properties.

d) Here we need the small detail that we used $\mathrm{id}(u)$ in Definition 3.9. Denote $\mathrm{ind}(q)$ by v. If both $\mathrm{fun}_1(v) \subseteq \eta$ and $\mathrm{fun}_2(v) \subseteq \nu$ hold, then the claim follows from (c).

Let $\eta', \nu' \in {}^{\mu}2$ be such that $\mathrm{fun}_1(v) \subseteq \eta'$ and $\mathrm{fun}_2(v) \subseteq \nu'$. Fix elements u^0, u^1, u^2 from U^2_λ so that

> u^0 is the smallest in $\triangleleft$-order with $\mathrm{fun}_1(u^0) \subseteq \eta$, $\mathrm{fun}_2(u^0) \subseteq \eta'$, and $\theta \subseteq \mathrm{dom}(p_{u^0})$;
>
> u^1 is the smallest in $\triangleleft$-order with $\mathrm{fun}_1(u^1) \subseteq \eta'$, $\mathrm{fun}_2(u^1) \subseteq \nu'$, $v \triangleleft u^1$ and $\mathrm{ran}(p_{u^0} \restriction \theta) \subseteq \mathrm{dom}(p_{u^1})$;
>
> u^2 is the smallest in $\triangleleft$-order with $\mathrm{fun}_1(u^2) \subseteq \nu'$, $\mathrm{fun}_2(u^2) \subseteq \nu$, $\mathrm{ran}(p_{u^1} \restriction \mathrm{ran}(p_{u^0} \restriction \theta)) \subseteq \mathrm{dom}(p_{u^2})$;

Define $\bar{f}$ to be $\bar{g}^{\bar{w},W}$, where $\bar{w} = \langle u^i \mid 0 \le i \le 2 \rangle$. Then $\bar{f}$ is in $\bar{\mathcal{F}}[\eta,\nu]$.

Define ξ_1 to be $\min\{\zeta + 1 \mid \zeta \in \mathrm{dom}(q) \text{ and } \eta(\zeta) \neq \mathrm{fun}_1(v)(\zeta)\}$, and ξ_2 to be $\min\{\zeta + 1 \mid \zeta \in \mathrm{dom}(q) \text{ and } \nu(\zeta) \neq \mathrm{fun}_2(v)(\zeta)\}$. Since $\mathrm{beg}(q) \subseteq \eta$ and $\mathrm{end}(q) \subset \nu$, we have that $\xi_1 \ge \sup^+(\mathrm{dom}(q))$ and $\xi_2 \ge \sup^+(\mathrm{ran}(q))$. So $\eta \restriction \xi_1 = \eta' \restriction \xi_1$ and $\nu' \restriction \xi_2 = \nu \restriction \xi_2$ ensure that ${f_0}^{-1} \restriction \mathrm{dom}(q)$ is identity and $f_2 \restriction \mathrm{ran}(q)$ is identity. Therefore $q \subseteq f$.

e) Since $\mathrm{dom}(f) \cup \mathrm{ran}(f)$ is bounded in μ it follows from Lemma 3.15(d) that $\mathrm{dom}(f_i) \cup \mathrm{ran}(f_i)$ are bounded in μ for all $i < \mathrm{lh}(\bar{f})$. Hence for every $i < \mathrm{lh}(\bar{f})$, $f_i \in V_\mu$, and by (a) we may assume, $\mathrm{ind}(f_i) \in V_\mu$. The claim follows from (d) by induction on $i < \mathrm{lh}(\bar{f})$. $\blacksquare_{3.20}$

4 The strongly equivalent non-isomorphic models

Recall that κ is a fixed strongly inaccessible cardinal and λ is a fixed regular cardinal below κ.

For ordinals $\theta < \mu$ and subsets A of μ, $[A]^\theta$ is the set of all θ-sequences of ordinals in A. For every $\theta < \mu < \kappa$ write $[\mu]^\theta_B$ for the set

$$\{\boldsymbol{a} \in [\mu]^\theta \mid \sup^+(\boldsymbol{a}) < \mu \text{ and for all } i < j < \theta, \boldsymbol{a}_i \neq \boldsymbol{a}_j\}$$

and denote $\bigcup_{\theta<\mu} [\mu]^\theta_B$ by $[\mu]^{<\mu}_B$. We write $\mathbf{0}$ for the function having domain κ and range $\{0\}$.

Definition 4.1 *For every $\mu \in D \cup \{\kappa\}$, $\theta < \mu$, and $\boldsymbol{a} \in [\mu]^\theta_B$ we define a family $\langle R^\eta_{\boldsymbol{a}} \mid \eta \in {}^\mu 2\rangle$ of relations (having arity θ) on μ as follows:*

relations $R^\eta_{\boldsymbol{a}}$, $\eta \in {}^\mu 2$, are the smallest subsets of $[\mu]^\theta$ having the properties:

if $\eta = \mathbf{0}{\restriction}\mu$ then $\boldsymbol{a} \in R^\eta_{\boldsymbol{a}}$;

if there is $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}{\restriction}\mu, \eta]$ with $\mathrm{dom}(f) = \boldsymbol{a}$, then $f(\boldsymbol{a}) \in R^\eta_{\boldsymbol{a}}$.

Suppose $\mu \in D \cup \{\kappa\}$. Define ρ_μ to be the vocabulary $\{R_{\boldsymbol{a}} \mid \boldsymbol{a} \in [\mu]^{<\mu}_B\}$ where each $R_{\boldsymbol{a}}$ is a relation symbol of arity $\mathrm{lh}(\boldsymbol{a})$. For every $\eta \in {}^\mu 2$, let $\mathcal{M}_\eta$ be the ρ_μ-structure with domain μ and interpretations $(R_{\boldsymbol{a}})^{\mathcal{M}_\eta} = R^\eta_{\boldsymbol{a}}$ for all $\boldsymbol{a} \in [\mu]^{<\mu}_B$. For all $\chi \in D \cap \mu$ and $A \subseteq \mu$, we write $\mathcal{M}^{\rho_\chi}_\eta{\restriction}A$ for the models having vocabulary ρ_χ, domain A, and interpretations $(R_{\boldsymbol{a}})^{\mathcal{M}^{\rho_\chi}_\eta{\restriction}A} = R^\eta_{\boldsymbol{a}} \cap [A]^{\mathrm{lh}(\boldsymbol{a})}$ for all $\boldsymbol{a} \in [\chi]^{<\chi}_B$.

Fact 4.2 *Assume $\mu \in D \cup \{\kappa\}$ and $\eta \in {}^\mu 2$.*

a) *For every $\boldsymbol{a} \in [\mu]^{<\mu}_B$, $R^\eta_{\boldsymbol{a}}$ is a subset of $[\mu]^{<\mu}_B$.*

b) *For all $\chi \in D \cap \mu$, $\mathcal{M}_{\eta\restriction\chi} = \mathcal{M}^{\rho_\chi}_\eta{\restriction}\chi$.*

Proof. a) Assume that for some $\boldsymbol{b} \in R^{\eta}_{\boldsymbol{a}}$, $\sup^{+}(\boldsymbol{b}) = \mu$. Then there should be $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \eta]$ with $\mathrm{dom}(f) = \boldsymbol{a}$ and $f(\boldsymbol{a}) = \boldsymbol{b}$ contrary to Lemma 3.15(e) and the fact $\sup^{+}(\boldsymbol{a}) < \mu$.

b) Abbreviate $\eta\restriction\chi$ by ν and let $\boldsymbol{a}$ be a sequence from $[\chi]^{<\chi}_{B}$. The interpretation $(R_{\boldsymbol{a}})^{\mathcal{M}_\nu} = R^{\nu}_{\boldsymbol{a}}$ is a subset of the interpretation $(R_{\boldsymbol{a}})^{\mathcal{M}^{\rho\chi}_{\eta}\restriction\chi}$ since $\bar{\mathcal{F}}[\mathbf{0}\restriction\chi, \nu] \subseteq \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \eta]$. Suppose $\boldsymbol{b} \in (R_{\boldsymbol{a}})^{\mathcal{M}^{\rho\chi}_{\eta}\restriction\chi}$ and let $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \eta]$ be such that $\mathrm{dom}(f) = \boldsymbol{a}$ and $f(\boldsymbol{a}) = \boldsymbol{b}$. By Lemma 3.20(a), we may assume $\mathrm{ind}(f_i) \in U[<\chi]$ for every $i < \mathrm{lh}(\bar{f})$. Consequently, $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu', \nu]$ and $\boldsymbol{b} \in R^{\nu}_{\boldsymbol{a}}$. ■

Fact 4.3 *Suppose $\mu \in D \cup \{\kappa\}$ and $\eta, \nu \in {}^{\mu}2$.*

a) *For all $v \in U$ with $\mathrm{fun}_1(v) \subseteq \eta$ and $\mathrm{fun}_2(v) \subseteq \nu$, the function p_v is a partial isomorphism from $\mathcal{M}_\eta$ into $\mathcal{M}_\nu$.*

b) *For every $\theta < \mu$ and $\boldsymbol{b} \neq \boldsymbol{c} \in [\mu]^{\theta}_{B}$, if there exist $\boldsymbol{a} \in [\mu]^{\theta}_{B}$ satisfying*

$$\mathcal{M}_\eta \models R_{\boldsymbol{a}}(\boldsymbol{b}) \text{ and } \mathcal{M}_\nu \models R_{\boldsymbol{a}}(\boldsymbol{c}),$$

then there is $\bar{f} \in \bar{\mathcal{F}}[\eta, \nu]$ with $f(\boldsymbol{b}) = \boldsymbol{c}$.

Proof. Both of these properties are direct consequences of Definition 4.1 and Fact 3.13. ■

Lemma 4.4 *For all $s, t \in {}^{\kappa}2$,*

a) *$s \sim_{\phi,P} t$ implies $\mathcal{M}_s \cong \mathcal{M}_t$ ($\sim_{\phi,P}$ is given in Definition 2.2), and*

b) *if $\mathcal{M}_s \cong \mathcal{M}_t$ then $s \sim_{\phi,P} t$.*

Proof. a) Suppose $s \sim_{\phi,P} t$, and let $r : \kappa \to 2$ be such that $\langle V_\kappa, \in, P, s, t, r\rangle \models \phi$. For every $\delta \in C' = C_{s,t,r} \cap D$ define u_δ to be the tuple $\langle s\restriction\delta, t\restriction\delta, r\restriction\delta, C_{s,t,r} \cap \delta\rangle$ ($C_{s,t,r}$ is given in Definition 3.2 and D is given in Definition 3.18). Directly by Definition 3.2, for all $\delta < \epsilon \in C'$, u_δ, u_ϵ are in U^1 and $u_\delta \lhd u_\epsilon$. Hence $p_{u_\delta} \subseteq p_{u_\epsilon}$ for $\delta < \epsilon \in C'$, and moreover, for the function $h = \bigcup_{\delta \in C'} p_{u_\delta}$ both of the equations $\mathrm{dom}(h) = \kappa$ and $\mathrm{ran}(h) = \kappa$ hold. Consequently h is an isomorphism from $\mathcal{M}_s$ onto $\mathcal{M}_t$.

b) Suppose $s \neq t$ and for fixed $\xi < \kappa$, $s(\xi) \neq t(\xi)$. Let h be an isomorphism from $\mathcal{M}_s$ onto $\mathcal{M}_t$, and let S' be the set $\{\delta \in \kappa \smallsetminus (\xi + 1) \mid$

$h[\delta] = \delta$ is a cardinal of cofinality $\geq \lambda\}$. Since h is an isomorphism and $s \restriction \delta \neq t \restriction \delta$ for all $\delta \in S'$, it follows from Fact 4.3(b) that for every $\delta \in S'$ there is a sequence $\bar{f}^\delta \in \bar{\mathcal{F}}[s,t]$ such that $f^\delta = h \restriction \delta$. For all $\delta < \epsilon \in S'$, $f^\delta = h \restriction \delta \subseteq h \restriction \epsilon = f^\epsilon$. Since S' is stationary in κ, there are $n < \omega$ and a stationary subset S of S' such that the equation $\mathrm{lh}(\bar{f}^\delta) = n$ holds for every $\delta \in S$.

Consider some $\delta \in S$ and $i < n$. Abbreviate $\mathrm{ind}(f_i^\delta)$ by u_i^δ. By Lemma 3.15(e), $\mathrm{dom}(f_i^\delta) = \mathrm{ran}(f_i^\delta) = \delta$. Moreover, by Lemma 3.17(c), $u_i^\delta \trianglelefteq u_i^\epsilon$ and $f_i^\delta \subseteq f_i^\epsilon$ for all $\epsilon \in S \setminus \delta$. By Fact 3.10(f), u_i^δ is in U^1. So $f_i^\delta = p_{u_i^\delta}$ and $\mathrm{dom}(f_i^\delta) = \delta = \mathrm{ord}(u_i^\delta) = \beth_\delta$. Define for each $i < n$,

$$\begin{aligned} s_i &= \textstyle\bigcup_{\delta \in S} \mathrm{fun}_1(u_i^\delta); \\ r_i &= \textstyle\bigcup_{\delta \in S} \mathrm{fun}_3(u_i^\delta); \end{aligned}$$

and let s_n be $\bigcup_{\delta \in S} \mathrm{fun}_2(u_{n-1}^\delta)$. Then $s = s_0$ and $t = s_n$.

We claim that $s \sim_{\phi,P} t$. By the transitivity of $\sim_{\phi,P}$ it is enough to show that for every $i < n$, r_i witness $s_i \sim_{\phi,P} s_{i+1}$. Contrary to this subclaim assume that for some $i < n$,

$$\langle V_\kappa, \in, P, s_i, s_{i+1}, r_i \rangle \not\models \phi.$$

Then there is $\delta \in S$ for which

$$\langle V_\delta, \in, P \cap V_\delta, s_i \restriction \delta, s_{i+1} \restriction \delta, r_i \restriction \delta \rangle \prec \langle V_\kappa, \in, P, s_i, s_{i+1}, r_i \rangle.$$

However $s_i \restriction \delta = \mathrm{fun}_1(u_i^\delta)$, $s_{i+1} \restriction \delta = \mathrm{fun}_2(u_i^\delta)$, and $r_i \restriction \delta = \mathrm{fun}_3(u_i^\delta)$, and so

$$\langle V_\delta, \in, P \cap V_\delta, \mathrm{fun}_1(u_i^\delta), \mathrm{fun}_2(u_i^\delta), \mathrm{fun}_3(u_i^\delta) \rangle \not\models \phi,$$

contrary to the fact that u_i^δ is in U^1. $\blacksquare_{4.4}$

In the following two lemmas we assume existence of a regular cardinal μ in D. Such μ does not necessarily exists, if κ is an arbitrary strongly inaccessible cardinal. However, these lemmas are only preliminaries for the main lemma, Lemma 4.7, where we assume κ to be a weakly compact cardinal. Note, when $\mu = \kappa$ in Lemma 4.5(a) below, it suffices that κ is a strongly inaccessible cardinal.

Lemma 4.5 *Suppose $\mu \in D$ is a regular cardinal or $\mu = \kappa$, and that η, ν are functions from μ into 2.*

a) $\mathcal{M}_\eta \equiv_{\infty\mu;\lambda} \mathcal{M}_\nu$.

b) *For every $\theta < \mu$, the model $\mathcal{M}_\eta$ satisfies the $L_{\infty\mu}$-sentence*

$$\forall \langle x_i \mid i < \theta\rangle \Big(\bigvee_{\boldsymbol{a} \in [\mu]^\theta_B} R_{\boldsymbol{a}}\big(\langle x_i \mid i < \theta\rangle\big)\Big)$$

c) *For all $\boldsymbol{a} \in [\mu]^{<\mu}_B$ and $\xi < \mu$, the following $L_{\infty\mu}$-sentence holds in $\mathcal{M}_\eta$:*

$$\forall \bar{x}\Big(R_{\boldsymbol{a}}(\bar{x}) \to \exists y \big(R_{\langle\xi\rangle \frown \boldsymbol{a}}(\langle y\rangle \frown \bar{x})\big)\Big).$$

d) *For all $\boldsymbol{a} \in [\mu]^{<\mu}_B$, $\mathcal{M}_\eta$ satisfies the $L_{\infty\mu}$-sentence:*

$$\forall \bar{x} \forall y \Big(R_{\boldsymbol{a}}(\bar{x}) \to \bigvee_{\xi<\mu} R_{\langle\xi\rangle \frown \boldsymbol{a}}(\langle y\rangle \frown \bar{x})\Big).$$

Proof. a) We give a winning strategy for player $\exists$ in the game $\mathrm{EF}_{\mu;\lambda}(\mathcal{M}_\eta, \mathcal{M}_\nu)$ (see Definition 2.1). Suppose $i < \lambda$ and for each $j \leq i$, player $\forall$ has chosen $X_j \in \{\mathcal{M}_\eta, \mathcal{M}_\nu\}$ and $A_j \subseteq \mu$ (where μ is the domain of both $\mathcal{M}_\eta$ and $\mathcal{M}_\nu$). Suppose that for every $j < i$, player $\exists$ has replied with a partial isomorphism p_{u_j} satisfying that $u^j \in U^2_\lambda$, $\mathrm{fun}_1(u^j) \subseteq \eta$, $\mathrm{fun}_2(u^j) \subseteq \nu$, $\bigcup_{k \leq j} A_k \subseteq \mathrm{dom}(p_{u_j}) \cap \mathrm{ran}(p_{u_j})$, and for all $k < j$, $u^k \lhd u^j$. Since $i < \lambda$ and $u^j \in U^2_\lambda$ for each $j < i$, the tuple $v = \bigcup_{j<i} u^j$ is in U^2_λ by Fact 3.6(a). Let θ be the smallest ordinal which is strictly greater than any ordinal in $\bigcup_{j \leq i} A_j$ ($\theta < \mu$ since μ is regular, $i < \mu$, and $\mathrm{card}(A_j) < \mu$ for every $j \leq i$). By Lemma 3.20(d), there is u^i in $U[<\mu]$ satisfying that $v \lhd u^i$, $\mathrm{fun}_1(u^i) \subseteq \eta$, $\mathrm{fun}_2(u_i) \subseteq \nu$, and $\theta \subseteq \mathrm{dom}(p_{u^i}) \cap \mathrm{ran}(p_{u^i})$. Since $\bigcup_{j<i} p_{u^j} = p_v \subseteq p_{u^i}$, the partial isomorphism p_{u^i} is a valid reply for player $\exists$ in the round i.

b) By Definition 4.1, for every $b \in [\mu]^\theta_B$, $R_{\boldsymbol{b}}(\boldsymbol{b})$ is satisfied in $\mathcal{M}_{\mathbf{0}\restriction\mu}$. The claim follows from (a).

c) By (a) we may assume $\eta = \mathbf{0}\restriction\mu$. For $\bar{x} = \boldsymbol{a}$ the claim holds directly by Definition 4.1. For any $\bar{x} = \boldsymbol{b} \in R^{\mathbf{0}}_{\boldsymbol{a}} \setminus \{\boldsymbol{a}\}$ there is some $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \mathbf{0}\restriction\mu]$ such that $\mathrm{dom}(f) = \boldsymbol{a}$ and $f(\boldsymbol{a}) = \boldsymbol{b}$. Since $\mu \in D$ there is by Lemma 3.20(e), $\bar{g} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \mathbf{0}\restriction\mu]$ with $g \supseteq f$ and $\mathrm{dom}(g) = \langle\xi\rangle \frown \boldsymbol{a}$.

d) Analogously to the proof of (c). If $\bar{x} = \boldsymbol{b}$ and $y = \zeta$ then there is some $\bar{f} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \mathbf{0}\restriction\mu]$ such that $f(\boldsymbol{a}) = \boldsymbol{b}$. Moreover by Lemma 3.20(e), there is $\bar{g} \in \bar{\mathcal{F}}[\mathbf{0}\restriction\mu, \mathbf{0}\restriction\mu]$ with $g \supseteq f$ and $\mathrm{ran}(g) = \langle\zeta\rangle \frown \boldsymbol{b}$. ■

Lemma 4.6 *Suppose μ is a regular cardinal in D and A is a subset of κ having cardinality μ.*

a) *Suppose $\eta \in {}^{\mu}2$, $A \subseteq \mu$, and $\mathcal{M}_{\eta}^{\rho\mu}\restriction A \equiv_{\infty\mu} \mathcal{M}_{\eta}$. Then $A = \mu$.*

b) *If $\mathcal{M}_{\mathbf{0}}^{\rho\mu}\restriction A \equiv_{\infty\mu} \mathcal{M}_{\mathbf{0}\restriction\mu}$ then there is $\eta \in {}^{\mu}2$ for which $\mathcal{M}_{\mathbf{0}}^{\rho\mu}\restriction A \cong \mathcal{M}_{\eta}$.*

Proof. a) Suppose, contrary to the claim, that $\xi < \mu$ is not in A. Let $\boldsymbol{b}$ in $[\mu]^{\omega}_{B}$ be such that $\boldsymbol{b} \subseteq A$, $\boldsymbol{b}_0 > \xi$ and for every $i < \omega$, $\boldsymbol{b}_i < \mathrm{card}(\boldsymbol{b}_{i+1})$. This is possible since $\mu = \mathrm{cf}(\mu)$, $\mu \in D$ implies μ is an uncountable limit cardinal, and $\mathrm{card}(A) = \mu$ implies that A is unbounded in μ. By Lemma 4.5(b), there is $\boldsymbol{a} \in [\mu]^{\omega}_{B}$ such that $\mathcal{M}_{\eta} \models R_{\boldsymbol{a}}(\boldsymbol{b})$. Since $\boldsymbol{b} \subseteq A$ also $\mathcal{M}_{\eta}^{\rho\mu}\restriction A$ satisfies $R_{\boldsymbol{a}}(\boldsymbol{b})$. By Lemma 4.5(d), there is $\xi' < \mu$ such that $\mathcal{M}_{\eta} \models R_{\langle\xi'\rangle\frown\boldsymbol{a}}(\langle\xi\rangle \frown \boldsymbol{b})$. By Lemma 4.5(c), there should be $\zeta \in A$ with $\mathcal{M}_{\eta}^{\rho\mu}\restriction A \models R_{\langle\xi'\rangle\frown\boldsymbol{a}}(\langle\zeta\rangle \frown \boldsymbol{b})$. However, then by Fact 4.3(b), there should be $\bar{f} \in \bar{\mathcal{F}}$ satisfying $\mathrm{dom}(f) = \{\xi\} \cup \boldsymbol{b}$, $f(\xi) = \zeta \neq \xi$, and $f(\boldsymbol{b}) = \boldsymbol{b}$ contrary to Lemma 3.17(d).

b) Our proof has the following structure:

> When $A \subseteq \mu$ the claim follows from (a) and Fact 4.2(b).
>
> The case that A is not a subset of $\zeta + \mu$ for any $\zeta \in A$ is shown to be impossible.
>
> Lastly we prove that when $A \subseteq \zeta + \mu$ for some $\zeta \in A$, there are $\eta \in {}^{\mu}2$ and $\bar{g} \in \bar{\mathcal{F}}$ such that $\mathrm{dom}(g) = \mu$, $\mathrm{ran}(g) = A$, $\mathrm{beg}(g) = \eta$, and $\mathrm{end}(g) \subseteq \mathbf{0}$. So g is an isomorphism between $\mathcal{M}_{\eta}$ and $\mathcal{M}_{\mathbf{0}}^{\rho\mu}\restriction A$.

Suppose there is an ω-sequence $\boldsymbol{b}$ such that $\boldsymbol{b}_0 > \mu$ and for all $l < \omega$, $\boldsymbol{b}_l \in A$ and $\boldsymbol{b}_{l+1} \geq \boldsymbol{b}_l + \mu$. By the equivalence $\mathcal{M}_{\mathbf{0}}^{\rho\mu}\restriction A \equiv_{\infty\mu} \mathcal{M}_{\mathbf{0}\restriction\mu}$ and Lemma 4.5(b), there is $\boldsymbol{a} \in [\mu]^{\omega}_{B}$ such that $\mathcal{M}_{\mathbf{0}}^{\rho\mu}\restriction A \models R_{\boldsymbol{a}}(\boldsymbol{b})$. Hence there should be $\bar{f} \in \bar{\mathcal{F}}$ with $\mathrm{dom}(f) = \boldsymbol{a}$ and $f(\boldsymbol{a}) = \boldsymbol{b}$ contrary to Lemma 3.16(f).

Suppose $\zeta \in A$ and $A \subseteq \zeta + \mu$. As above, there are $\bar{f} \in \bar{\mathcal{F}}$ and $\gamma < \mu$ with $\mathrm{dom}(f) = \{\gamma\}$, $f(\gamma) = \zeta$, $\mathrm{beg}(f) = \mathbf{0}\restriction(\gamma + 1)$, and $\mathrm{end}(f) = \mathbf{0}\restriction(\zeta + 1)$. Since $\gamma < \mu \leq \zeta = f(\gamma)$, there is the smallest index $k < \mathrm{lh}(\bar{f})$ such that $f_{<k}(\gamma) < \mu$ and $f_{\leq k}(\gamma) \geq \mu$. By Lemma 3.15(c), f_j is increasing for all $j \in \{k, \ldots, \mathrm{lh}(\bar{f}) - 1\}$. Let $\bar{u}$ be the sequence $\langle \mathrm{ind}(f_j) \mid j \in \{k, \ldots, \mathrm{lh}(\bar{f}) - 1\}\rangle$. By Lemma 3.16(e), the sequence $\bar{g}^{\bar{u},\mu}$ is a well-defined member of $\bar{\mathcal{F}}$, and moreover, $\mathrm{end}(g^{\bar{u},\mu}) \subseteq \mathbf{0}$. Abbreviate this sequence by g and the ordinal $f_{<k}(\gamma)$ by ξ. We define the wanted η to be $\mathrm{beg}(g)$.

Finally we show that for this g the equation $A = \mathrm{ran}(g)$ holds. Suppose ζ' is in A but not in $\mathrm{ran}(g)$. By the equivalence and Lemma 4.5(b), there are $\epsilon, \epsilon' < \mu$ such that $R_{\langle\epsilon\rangle}(\zeta)$ and $R_{\langle\epsilon',\epsilon\rangle}(\zeta',\zeta)$ hold in $\mathcal{M}_{\mathbf{0}}^{\rho_\mu}\restriction A$. By Lemma 4.5(c), there is $\xi' < \mu$ for which $R_{\langle\epsilon',\epsilon\rangle}(\xi',\xi)$ in $\mathcal{M}_\eta$. However, by Fact 4.3(b), there should be $\bar{h} \in \bar{\mathcal{F}}$ with $h(\xi) = \zeta$ and $h(\xi') = \zeta'$, contrary to Lemma 3.16(g). From another direction, if $A \subsetneqq \mathrm{ran}(g)$, then η and the set $B = g^{-1}[A] \subsetneqq \mu$ contradict (a), since by Lemma 4.5(a), $\mathcal{M}_\eta \equiv_{\infty\mu} \mathcal{M}_{\mathbf{0}\restriction\mu}$, by our assumption, $\mathcal{M}_{\mathbf{0}\restriction\mu} \equiv_{\infty\mu} \mathcal{M}_{\mathbf{0}}^{\rho_\mu}\restriction A$, and $g^{-1}\restriction A : \mathcal{M}_{\mathbf{0}}^{\rho_\mu}\restriction A \cong \mathcal{M}_\eta^{\rho_\mu}\restriction B$. ■4.6

Lemma 4.7 *Suppose κ is a weakly compact cardinal and $\mathcal{M}$ is a model of cardinality κ with $\mathcal{M} \equiv_{\infty\kappa} \mathcal{M}_{\mathbf{0}}$. Then there is $s \in {}^\kappa 2$ for which $\mathcal{M} \cong \mathcal{M}_s$.*

Proof. Without loss of generality we may assume that the domain of $\mathcal{M}$ is κ. By the $\equiv_{\infty\kappa}$-equivalence of the models $\mathcal{M}$ and $\mathcal{M}_{\mathbf{0}}$, let for every regular cardinal $\mu < \kappa$, A_μ be a subset of κ such that $\mathcal{M}^{\rho_\mu}\restriction\mu \cong \mathcal{M}_{\mathbf{0}}^{\rho_\mu}\restriction A_\mu$. Let Y be the set given in Definition 3.18. Note that for all $\mu \in D \cup \{\kappa\}$ and $\eta \in {}^\mu 2$, the model $\mathcal{M}_\eta$ is definable from η and $Y \cap V_\mu$. Let τ be a winning strategy for $\exists$ in the game $\mathrm{EF}_{\kappa;\omega}(\mathcal{M}, \mathcal{M}_{\mathbf{0}})$. Assume now, contrary to the claim, that $\mathcal{M} \ncong \mathcal{M}_s$ for all $s \in {}^\kappa 2$. Because κ is Π^1_1-indescribable, there is a regular cardinal $\mu < \kappa$ such that $\langle V_\mu, \in, \mathcal{M}^{\rho_\mu}\restriction\mu, \tau \cap V_\mu, Y \cap V_\mu\rangle$ satisfies the following:

$$\text{for all } \eta \in {}^\mu 2, \mathcal{M}^{\rho_\mu}\restriction\mu \ncong \mathcal{M}_\eta.$$

Then $\mathcal{M}_{\mathbf{0}\restriction\mu} = \mathcal{M}_{\mathbf{0}}^{\rho_\mu}\restriction\mu \equiv_{\infty\mu} \mathcal{M}^{\rho_\mu}\restriction\mu$, and by the isomorphism $\mathcal{M}^{\rho_\mu}\restriction\mu \cong \mathcal{M}_{\mathbf{0}}^{\rho_\mu}\restriction A_\mu$, we have $\mathcal{M}_{\mathbf{0}\restriction\mu} \equiv_{\infty\mu} \mathcal{M}_{\mathbf{0}}^{\rho_\mu}\restriction A_\mu$ and for all $\eta \in {}^\mu 2$, $\mathcal{M}_{\mathbf{0}}^{\rho_\mu}\restriction A_\mu \ncong \mathcal{M}_\eta$. This contradicts Lemma 4.6(b). ■

Lemma 4.8 *Suppose κ is a weakly compact cardinal, $\lambda < \kappa$ is a regular cardinal, and there is a Σ^1_1-equivalence relation on ${}^\kappa 2$ having μ different equivalent classes. Then there exists a model $\mathcal{M}$ such that the vocabulary of $\mathcal{M}$ consists of one relation symbol of finite arity,* $\mathrm{card}(\mathcal{M}) = \kappa$, *and* $\mathrm{No}_\lambda(\mathcal{M}) = \mu$.

Proof. By the preceding lemmas the model $\mathcal{M}_{\mathbf{0}}$ defined as in Definition 4.1 satisfies the claim, except that the vocabulary of $\mathcal{M}$ is overly large. However, by [She85, Claim 1.3(1)], the inaccessibility of κ ensures that there is a model $\mathcal{N}$ of cardinality κ with λ many relations of finite arity satisfying $\mathrm{No}(N) = \mathrm{No}(\mathcal{M}_{\mathbf{0}})$ (the proof is a simple coding). Furthermore, by [She85, Claim 1.4(2)], the λ many relations can be coded by one relation so that the other properties are preserved. Actually, the claims cited concern the

case $\lambda = \aleph_0$, but there is no problem to preserve $\mathrm{No}_\lambda(\mathcal{M}_{\mathbf{0}})$ in the cases $\aleph_0 < \lambda < \kappa$, too. $\blacksquare_{4.8}$

References

[Cha68] C. C. Chang. Some remarks on the model theory of infinitary languages. In J. Barwise, editor, *The syntax and semantics of infinitary languages*, Lecture Notes in Math., 72, pages 36–63. Springer-Verlag, Berlin, 1968.

[Hyt90] Tapani Hyttinen. Model theory for infinite quantifier languages. *Fund. Math.*, 134(2):125–142, 1990.

[Kar65] Carol R. Karp. Finite-quantifier equivalence. In *Theory of Models (Proc. 1963 Internat. Sympos. Berkeley)*, pages 407–412. North-Holland, Amsterdam, 1965.

[Kar84] Maaret Karttunen. Model theory for infinitely deep languages. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (50):96, 1984.

[Oik97] Juha Oikkonen. Undefinability of κ-well-orderings in $L_{\infty\kappa}$. *J. Symbolic Logic*, 62(3):999–1020, 1997.

[Pal77a] E. A. Palyutin. Number of models in L_{∞,ω_1} theories, II. *Algebra i Logika*, 16(4):443–456, 1977. English translation in [Pal77b].

[Pal77b] E. A. Palyutin. Number of models in L_{∞,ω_1} theories, II. *Algebra and Logic*, 16(4):299–309, 1977.

[Sco65] Dana Scott. Logic with denumerably long formulas and finite strings of quantifiers. In J. W. Addison, Leon Henkin, and Alfred Tarski, editors, *Theory of Models (Proc. 1963 Internat. Sympos. Berkeley)*, pages 329–34. North-Holland, Amsterdam, 1965.

[She81] Saharon Shelah. On the number of nonisomorphic models of cardinality λ $L_{\infty,\lambda}$-equivalent to a fixed model. *Notre Dame J. Formal Logic*, 22(1):5–10, 1981.

[She82] Saharon Shelah. On the number of nonisomorphic models in $L_{\infty,\lambda}$ when κ is weakly compact. *Notre Dame J. Formal Logic*, 23(1):21–26, 1982.

[She85] Saharon Shelah. On the possible number no(M) = the number of nonisomorphic models $L_{\infty,\lambda}$-equivalent to M of power λ, for λ singular. *Notre Dame J. Formal Logic*, 26(1):36–50, 1985.

[She86] Saharon Shelah. On the no(M) for M of singular power. In *Around classification theory of models*, Lecture Notes in Math., 1182, pages 120–134. Springer-Verlag, Berlin, 1986.

[SVa] Saharon Shelah and Pauli Väisänen. On equivalence relations σ^1_1-definable over $h(\kappa)$. No. 719 in the list of Shelah's publications.

[SVb] Saharon Shelah and Pauli Väisänen. On inverse γ-systems and the number of $L_{\infty,\lambda}$-equivalent, non-isomorphic models for λ singular. To appear in J. Symbolic Logic. No. 644 in the list of Shelah's publications.

[SVc] Saharon Shelah and Pauli Väisänen. On the number of L_{∞,ω_1}-equivalent, non-isomorphic models. To appear in Trans. Amer. Math. Soc.. No. 646 in the list of Shelah's publications.

Saharon Shelah:

Institute of Mathematics
The Hebrew University
Jerusalem. Israel

Rutgers University
Hill Ctr-Busch
New Brunswick. New Jersey 08903
`shelah@math.rutgers.edu`

Pauli Väisänen:

Department of Mathematics
P.O. Box 4
00014 University of Helsinki
Finland
`pauli.vaisanen@helsinki.fi`